Spacetime Constraints

(Exact Control)

Byeong-Chun SHIN

November 24, 1999

Define the $L^2(0,1)$-norm $||\cdot||$ by

$$||f||^2 := \int_0^1 |f(t)|^2 \, dt.$$

A Minimization Problem :

Find a control $f \in L^2(0,1)$ which minimizes

$$||f||^2$$

subject to the constraints

$$x''(t) + x'(t) = f(t) \tag{1}$$

$$x(0) = x'(0) = 0, \quad x(1) = 1, \quad x'(1) = 0.$$

Step 1 : Find the homo. sols. for Eq. (1).

$$\lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda = 0, -1.$$

$$\text{Basis functions :} \quad x_1 = 1, \quad x_2 = \mathrm{e}^{-t}.$$

$$\text{Homogeneous Sol. :} \quad x_h(t) = C_1 + C_2\mathrm{e}^{-t}.$$

Step 2 : Find a particular sol. for Eq. (1).

$$\text{Wronskian :} \qquad w(t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = -\mathrm{e}^{-t}.$$

A particular solution is

$$\begin{aligned} x_p(t) &= -x_1 \int_0^t \frac{x_2\ f}{w}\, du + x_2 \int_0^t \frac{x_1\ f}{w}\, du \\ &= \int_0^t f(u)\, du - \mathrm{e}^{-t} \int_0^t \mathrm{e}^u\, f(u)\, du. \end{aligned}$$

Step 3 : General Solution :

$$x(t) = x_h(t) + x_p(t).$$

Step 4 : Initial Conditions :

$$x(0) = x'(0) = 0 \quad \Rightarrow \quad C_1 = C_2 = 0.$$

$$x(t) = x_p(t).$$

Step 5 : Check the constraints :

$$x(1) = 1, \quad x'(1) = 0$$

$$\text{with} \quad x'(t) = \mathrm{e}^{-t} \int_0^t \mathrm{e}^u f(u)\, du.$$

Hence

$$x(1) = \int_0^1 \phi_1(t) f(t)\, dt = 1$$

$$x'(1) = \int_0^1 \phi_2(t) f(t)\, dt = 0,$$

that is,

$$(\phi_1 \mid f) = 1, \quad (\phi_2 \mid f) = 0$$

where

$$\phi_1(t) = 1 - \mathrm{e}^{t-1}, \quad \phi_2(t) = \mathrm{e}^{t-1}.$$

Final Step 6 :

Find a control $f = a_1\phi_1 + a_2\phi_2 \in M$ such that

$$(\phi_1 \mid f) = 1, \quad (\phi_2 \mid f) = 0.$$

The matrix form :

$$A\,\mathbf{a} = \mathbf{b},$$

where

$$A_{ii} := (\phi_i \mid \phi_i), \quad i = 1, 2,$$

$$A_{12} = A_{21} := (\phi_1 \mid \phi_2),$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Proof of the Final Step

From the **Step 5**, we have

Changed Minimization Problem(2) :

Find a control $f \in L^2(0,1)$ which minimizes

$$||f||^2$$

subject to the constraints

$$(\phi_1 \mid f) = 1, \quad (\phi_2 \mid f) = 0 \tag{2}$$

Choose a $g \in L^2(0,1)$ which satisfies Eq. (2):

$$(\phi_1 \mid g) = 1, \quad (\phi_2 \mid g) = 0.$$

Define a space

$$M := \mathrm{Span}\{\phi_1,\ \phi_2\}$$

and let $M^{\perp}$ be its orthogonal complement.

Let $h = g - f$, i.e., $f = g - h$.

Then $h \in M^{\perp}$, i.e.,

$$(\phi_1 \mid h) = 0, \quad (\phi_2 \mid h) = 0.$$

Changed Minimization Problem(3) :

Find a control $f = g - h \in L^2(0,1)$ which minimizes

$$||g - h||^2, \quad h \in M^{\perp}.$$

This is equivalent to

Find the orthogoanl projection h_g over $M^{\perp}$,

$$f = g - h_g \quad \perp \quad M^{\perp}.$$

So,

$$f = a_1\phi_1 + a_2\phi_2 \in M.$$

Changed Minimization Problem(4) :

Find a control $f = a_1\phi_1 + a_2\phi_2 \in M$ such that

$$(\phi_1 \mid f) = 1, \quad (\phi_2 \mid f) = 0.$$

The matrix form :

$$A\,\mathbf{a} = \mathbf{b},$$

where

$$A_{ii} := (\phi_i \mid \phi_i), \quad i = 1, 2,$$

$$A_{12} = A_{21} := (\phi_1 \mid \phi_2),$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$A_{11} = -\frac{1}{2}(1 - 4\mathrm{e}^{-1} + \mathrm{e}^{-2})$$
$$A_{22} = \frac{1}{2}(1 - \mathrm{e}^{-2})$$
$$A_{12} = A_{21} = \frac{1}{2}(1 - 2\mathrm{e}^{-1} + \mathrm{e}^{-2}).$$

The exact control $f(t)$ is

$$f(t) = \frac{1 + \mathrm{e} - 2\mathrm{e}^{t}}{3 - \mathrm{e}}.$$