< Polynomial Fitting >

$p(x) = p_1 x^n + p_2 x^{n-1} + \cdots + p_n x + p_{n+1}$

$>> \; p = [p_1 \;\; p_2 \; \cdots \; p_n \;\; p_{n+1}]$

$>> \; x = -1 : 0.1 : 1;$

$>> \; y = \text{polyval}(p, x)$

If x is a matrix, then

$>> \; X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix};$

$>> \; Y = \text{polyvalm}(p, X)$

$$Y = p_1 X^n + p_2 X^{n-1} + \cdots + p_n X + p_{n+1} I.$$

< Multiplication and Division >

$>> \; g = [1 \;\; -6 \;\; 12 \;\; -8];$ $\quad h = [1 \;\; -2];$

$>> \; [q, r] = \text{deconv}(g, h)$: Division

$>> \; \text{conv}(h, q)$: Mulitiplication

Data :

$$\begin{cases} x_i : & x_1 & x_2 & \cdots & x_{m-1} & x_m \\ y_i : & y_1 & y_2 & \cdots & y_{m-1} & y_m \end{cases}$$

We want to find a poly. $p(x)$ s.t.

$p(x_i) = y_i$ for all $i = 1, 2, \cdots, m$.

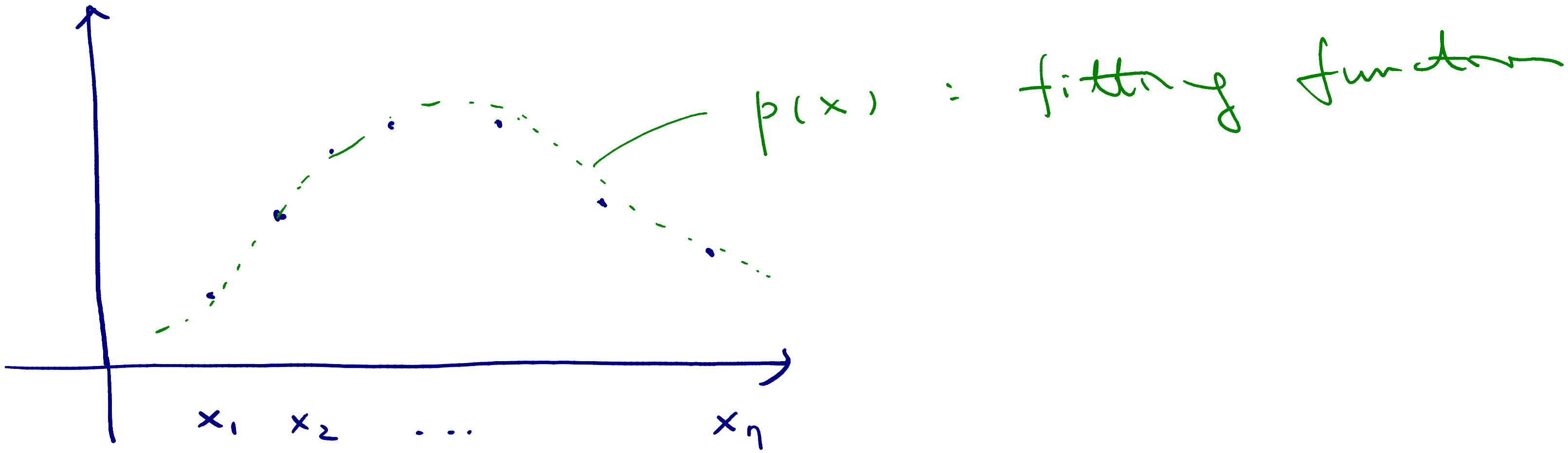
$\longrightarrow$ Lagrange interpolating polynomial of degree $\underline{n = m-1}$.

If $n < m-1$, then we can not find such poly. p satisfying $p(x_i) = y_i$.

Find $p(x)$ of degree $n < m-1$ which minimizes

$$\underline{G(p) = \sum_{i=1}^{m} | p(x_i) - y_i |^2}$$

>> p = polyfit(x, y, n).



① polyfit :

```
>> p = polyfit(x, y, n)
>> xi = linspace(a, b, k); % Grid points.
>> yi = polyval(p, xi); % Interpolation
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② interp1

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>> x = [0 pi/4 3*pi/8 3*pi/4 pi];
>> y = sin(x);
>> xi = linspace(0, pi, 40);
>> ys = interp1(x, y, xi, 'spline');
>> yl = interp1(x, y, xi, 'linear');
>> plot(x, y, 'o', xi, sin(xi), xi, ys, xi, yl);
```

< Two dimensional Interpolation >

$$\underbrace{zi}_{\text{matrix}} = \text{griddata}\,(\underbrace{x, y, z}_{\text{vectors}},\ \underbrace{xi, yi}_{\text{matrices}});$$

$\Longrightarrow$ mesh(xi, yi, zi).

```
> x = rand(100, 1)*4 - 2;   y = x;
> z = x.*exp(-x.^2 - y.^2);
> hi = -2 : 0.1 : 2;
> [xi, yi] = meshgrid(hi);
> zi = griddata(x, y, z, xi, yi);
> mesh(xi, yi, zi); hold
> plot3(x, y, z, 'o'); hold off
```

< Least-squares with special functions >

Let $x = (x_i)_m$, $y = (y_i)_m$ be a data and

let $\mathcal{B} = \text{span}(\phi_1, \phi_2, \cdots, \phi_n)$ be a function space based on some functions ϕ_i'.

Find a function $g \in \mathcal{B}$:

$$g(x) = \sum_{k=1}^{n} a_k \phi_k = a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots + a_n \phi_n(x)$$

which minimizes

$$F(g) = \sum_{i=1}^{m} |g(x_i) - y_i|^2 .$$

* g can be a function of $a_1, a_2, \cdots, a_n$
* $F(g)$ is indeed a function of $a_1, a_2, \cdots, a_n$
* That is, the unknowns are $a_1, a_2, \cdots, a_n$

$$F(g) = \sum_{i=1}^{m} |g(x_i) - y_i|^2$$

$$\Rightarrow F(a_1, \cdots, a_n) = \sum_{i=1}^{m} \left| \sum_{k=1}^{n} a_k \phi_k(x_i) - y_i \right|^2$$

From $\dfrac{\partial F}{\partial a_j} = 0$ for $j = 1, 2, \cdots, n$

$$\cancel{2} \sum_{i=1}^{m} \left(\sum_{k=1}^{n} a_k \phi_k(x_i) - y_i \right) \phi_j(x_i) = 0$$

$$\Rightarrow \sum_{k=1}^{n} a_k \left(\sum_{i=1}^{m} \phi_k(x_i) \phi_j(x_i) \right) = \sum_{i=1}^{m} \phi_j(x_i) y_i .$$

Hence, the j-th row is given by

$$\left[\sum_{i=1}^{m} \phi_j(x_i)\phi_1(x_i) \quad \sum_{i=1}^{m} \phi_j(x_i)\phi_2(x_i) \quad \cdots \quad \sum_{i=1}^{m} \phi_j(x_i)\phi_n(x_i) \right] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \left[\sum_{i=1}^{m} \phi_j(x_i) y_i \right]$$

Hence the system is given by

$$\begin{bmatrix} \sum_{i=1}^{m} \phi_1(x_i)\phi_1(x_i) & \sum_{i=1}^{m} \phi_1(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{m} \phi_1(x_i)\phi_n(x_i) \\ \sum_{i=1}^{m} \phi_2(x_i)\phi_1(x_i) & \sum_{i=1}^{m} \phi_2(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{m} \phi_2(x_i)\phi_n(x_i) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} \phi_n(x_i)\phi_1(x_i) & \sum_{i=1}^{m} \phi_n(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{m} \phi_n(x_i)\phi_n(x_i) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} \phi_1(x_i) y_i \\ \sum_{i=1}^{m} \phi_2(x_i) y_i \\ \vdots \\ \sum_{i=1}^{m} \phi_n(x_i) y_i \end{bmatrix}$$

$A^T A u = A^T Y$: normal equation

$$\underbrace{\begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_m) & \phi_2(x_m) & \cdots & \phi_n(x_m) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{Y}$$

$A_{ij} = \phi_j(x_i)$: $m \times n$ matrix.

<Example>

Find a fitting function

$$g(x) = a_1 \sin(\pi x) + a_2 \cos(\pi x) + a_3 \pi x$$

on $[0, 1]$

for the data

$$\left(\frac{1}{6}, 1\right), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, -1\right), \left(\frac{3}{4}, 0\right).$$

Let $\phi_1 = \sin \pi(x)$, $\phi_2 = \cos \pi x$, $\phi_3 = \pi x$.

$$A = (A_{ij}) = \phi_j(x_i).$$

$$U = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \phi_1(X) & \phi_2(X) & \phi_3(X) \end{bmatrix}$$