Addendum to "The orbit of a β-transformation cannot lie in a small interval, J. Korean Math. Soc. 49 (2012), no. 4, 867-879"

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Abstract

The paper mentioned in the title is completed by this Addendum.

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Let $A = \{a_1, a_2, \ldots, a_n\}$ be a fixed real alphabet with $a_1 < a_2 < \ldots < a_n$, and $\beta > 1$ a real number. We say that an infinite word $(x_i)_{i=1}^{\infty}$ over A is an *expansion* of x in *base* β if

$$x = \sum_{i=1}^{\infty} \frac{x_i}{\beta^i}$$

holds. If β is not an integer, then the expansions are no more unique in general. The uniqueness of expansions changes drastically as the base β varies. It turned out that there is a kind of boundary between uniqueness and non-uniqueness of expansions. At first, one readily notes that both $a_1^{\infty} := a_1 a_1 \cdots$ and a_n^{∞} are unique expansions in any base $\beta > 1$. It was proved in [1] that there exists a real number $\beta_0 > 1$ satisfying the following:

(i) if $1 < \beta < \beta_0$, then every $x \in \left(\frac{a_1}{\beta-1}, \frac{a_n}{\beta-1}\right)$ has multiple expansions in base β,

(ii) if $\beta > \beta_0$, then some $x \in \left(\frac{a_1}{\beta-1}, \frac{a_n}{\beta-1}\right)$ has a unique expansion in base β.

The number β_0 is called the *critical base* or *generalized golden ratio* for A. The current state of the art is very far from a complete characterization of critical bases for general alphabets A. Recently, the present author recognized that the work in [2] may play a key role when the alphabet A consists of three letters. He gave in [3] a procedure to determine the critical bases for ternary alphabets. An effective and algebraic algorithm was possible thanks to Theorem 4.7 and 4.8 of [2]. A computer program implementing the main idea was presented. And this program revealed a missing case in the theorems. To the previous twelve cases in Theorem 4.7 and 4.8, must we add another new case at Theorem 4.8 (c) (iv). The rest of this Addendum will make complete the paper mentioned in the title.

Theorem 4.8 $\cdots$

$\vdots$

(c) $\cdots$

$\vdots$

(iv) *If $ab < xy < ba$, then I contains the orbit closure of $((bua)^\omega)_\beta$, whose frequency is $a + \frac{|u|_b+1}{|u|+2}$.*

Proof. We have

$$as = auxyvs' < (aub)^\omega < (bua)^\omega < buxyvs' = bs.$$

□

References

[1] V. Komornik, A.C. Lai, and M. Pedicini. Generalized golden ratios of ternary alphabets. *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 4, 1113-1146.

[2] D.Y. Kwon. The orbit of a β-transformation cannot lie in a small interval. *J. Korean Math. Soc.* **49** (2012), no. 4, 867-879.

[3] D.Y. Kwon. Sturmian words and Cantor sets arising from unique expansions over ternary alphabets. *preprint* (2016).

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