Theorem 3.2. *Let $\alpha > 0$ and $|z| < 1$. Then*

$$\begin{aligned}\sum_{n=1}^{\infty}(\lceil \alpha n\rceil - \lceil \alpha(n-1)\rceil)z^n &= z + \sum_{\substack{0<p/q<\alpha\\ \gcd(p,q)=1}} \frac{z^q - z^{q+1}}{1-z^q}\\ &= z + \sum_{n=1}^{\infty}\varphi_\alpha(n)\frac{z^n - z^{n+1}}{1-z^n}.\end{aligned}$$

In particular,

$$z + \sum_{\substack{0<p/q<1\\ \gcd(p,q)=1}} \frac{z^q - z^{q+1}}{1-z^q} = \sum_{n=1}^{\infty}\varphi(n)\frac{z^n - z^{n+1}}{1-z^n} = \frac{z}{1-z}.$$

Before proving Theorem 3.1, we first notice that the summation in (1) converges absolutely whenever $\mathrm{Re}(s) > 1$.

Lemma 3.3. *We have* $s=\sigma+it$

$$|\zeta(s) - \zeta(s, 1+n^{-1})| \le \frac{|s|}{n}\zeta(\cancel{|s|}\,\sigma + 1).$$

Proof. One notes that

$$\zeta(s) - \zeta(s, 1+n^{-1}) = \sum_{k=1}^{\infty}\left(\frac{1}{k^s} - \frac{1}{(k+n^{-1})^s}\right) = \sum_{k=1}^{\infty} s\int_k^{k+n^{-1}} u^{-s-1}\,du,$$

which is followed by

$$\begin{aligned}|\zeta(s) - \zeta(s, 1+n^{-1})| &\le \sum_{k=1}^{\infty}|s|\int_k^{k+n^{-1}} u^{-\cancel{|s|}\sigma-1}\,du\\ &\le \sum_{k=1}^{\infty}|s|\int_k^{k+n^{-1}} k^{-\cancel{|s|}\sigma-1}\,du\\ &= \sum_{k=1}^{\infty}|s|\frac{1}{k^{\cancel{|s|}\sigma+1}}\frac{1}{n} = \frac{|s|}{n}\zeta(\cancel{|s|}\,\sigma+1).\end{aligned}$$

□