A one-parameter family of Dirichlet series whose coefficients are Sturmian words

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Abstract

Given $\alpha > 0$, let $(s_\alpha(n))_{n\geq 1}$ be the lexicographically greatest Sturmian word of slope α. We study Dirichlet series with $s_\alpha(n)$ coefficients. Its continuity and differentiability in α are investigated. As a consequence, we obtain another kind of singular function whose differentiability rests upon Diophantine approximation.

Keywords: Dirichlet series, singular function, Sturmian word
2010 MSC: 11M41, 11J82, 26A30, 68R15

1. Introduction

Let $\alpha \geq 0$ and $\rho \in [0, 1]$. Arithmetic functions $s_{\alpha,\rho}, s'_{\alpha,\rho}$, defined by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n + 1) + \rho\rfloor - \lfloor \alpha n + \rho\rfloor,$$

$$s'_{\alpha,\rho}(n) := \lceil \alpha(n + 1) + \rho\rceil - \lceil \alpha n + \rho\rceil,$$

bring forth two infinite words $s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots$ and $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$ over the alphabet $\{\lceil\alpha\rceil - 1, \lceil\alpha\rceil\}$. As usual, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, respectively. In particular, if α is an integer, then $s_{\alpha,\rho} = s'_{\alpha,\rho} = \alpha^\omega := \alpha\alpha\cdots$ for any ρ. Otherwise, both $\lceil\alpha\rceil - 1$ and $\lceil\alpha\rceil$ appear indefinitely often in $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$. The word $s_{\alpha,\rho}$ (resp. $s'_{\alpha,\rho}$) is called a *lower* (resp. *upper*) *mechanical word* with *slope* α and *intercept* ρ. Since the first introduction in [18], these words have enjoyed much attention from combinatorists on words. The interested readers are referred to [16] and

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the bibliography therein. *Sturmian words* are mechanical words of irrational slopes, but also used for general mechanical words in the literature. The title of the present paper reflects this abuse of terminology.

For typographical convenience, let us write, for $n \geq 1$,

$$s_\alpha(n) := \lceil \alpha n \rceil - \lceil \alpha(n-1) \rceil.$$

Then $s_\alpha := s_\alpha(1)s_\alpha(2)\cdots$ is nothing but $s'_{\alpha,0}$, and well known to be greatest in lexicographic order amongst all mechanical words of slope α [2].

This paper is the first consideration of Dirichlet series with Sturmian coefficients. But the coefficients are restricted to the case of s_α rather than general $s_{\alpha,\rho}, s'_{\alpha,\rho}$, i.e., we explore Dirichlet series of the form

$$Z(\alpha, s) := \sum_{n=1}^{\infty} \frac{s_\alpha(n)}{n^s}.$$

The reason for this restriction is twofold. First, $Z(\alpha, s)$ converges to the Riemann zeta function as α tends to 1 from the left. The second reason is rather subtle. Analysis for $Z(\alpha, s)$ appeals to Diophantine approximations. The case of $s_{\alpha,\rho}, s'_{\alpha,\rho}$ is, however, necessarily accompanied by inhomogeneous Diophantine approximations unless $\rho \in \alpha\mathbb{Z} + \mathbb{Z}$, which are much more sophisticated than homogeneous ones.

Similar kinds of Dirichlet series were investigated in two respects. Suppose that $\alpha > 0$ is irrational. Then Sturmian words of slope α are i) of lowest possible complexity among aperiodic words, and ii) almost periodic [18]. Dirichlet series with another low-complexity coefficients were examined by Allouche et al. [1]. Their coefficients are generated by finite automata. On the other hand, Knill and Lesieutre [9] considered Dirichlet series with almost periodic coefficients, of the form $\sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s}$ for a fixed irrational α and a 1-periodic functions g. Under a Diophantine assumption on α and for a real analytic periodic function g, they analytically continued $\sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s}$ to the entire complex plane. Let $0 < \alpha < 1$, and define a 1-periodic function g_α by

$$g_\alpha(x) := \begin{cases} 1, & 0 < x \leq \alpha, \\ 0, & \alpha < x \leq 1. \end{cases}$$

One notes then that $g_\alpha(n\alpha) = g_\alpha(\{n\alpha\}) = s_\alpha(n)$. Here, $\{\cdot\}$ denotes the fractional part. The present theme in this paper is reminiscent of those studied by Hecke [7] and by Hardy and Littlewood [5]. They investigated the Dirichlet series $\sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s}$ when $g(x) := \{x\} - 1/2$ and α is a quadratic irrational.

The papers mentioned above established analytic or meromorphic continuations to the entire complex plane. Such continuation of $Z(\alpha, s)$ for fixed α will be pursued in a subsequent work. The current concern is, though, in a disparate context.

2. Sturmian Dirichlet series and power series.

While Sturmian Dirichlet series is the major issue now, it is worthwhile to mention Sturmian power series

$$\Xi(\alpha, z) := \sum_{n=1}^{\infty} s_\alpha(n) z^n,$$

which was scrutinized from diverse angles by the author. This power series emerged, in an unexpected manner, from dynamics [10, 11, 12].

For fixed $0 < y_0 < 1$, the function $\Xi(x, y_0)$ in real x is singular, i.e., its derivative vanishes almost everywhere [13, 14]. Furthermore, at almost every point, we can determine whether $\Xi(x, y_0)$ is differentiable or not. At every positive algebraic irrational, e.g., the derivative exists and is equal to zero. In contrast, only qualitative statements are possible as to the differentiability of other famous singular functions such as Minkowski's $?(x)$ function [19, 3] and Riesz-Nágy function [22, 20]. Not a single point is known where we can determine whether such function is differentiable or not. Apart from the Cantor function whose derivative check is trivial, $\Xi(x, y_0)$ is the first ever singular function whose derivative can be manifested explicitly at some points. This benefit is also shared by Sturmian Dirichlet series as we will see below. While the derivative of $\Xi(x, y_0)$ entails very unusual Diophantine property, the one of Sturmian Dirichlet series involves usual irrationality measures.

In the context of Sturmian power series, Szegö's Theorem (see, e.g., [21]) reads as follows. If $\alpha > 0$ is rational, then $\Xi(\alpha, z)$ is a rational function. And otherwise, $\Xi(\alpha, z)$ has the unit circle as a natural boundary. On the other hand, the abscissa of convergence of Sturmian Dirichlet series can be easily computed. On general theory of Dirichlet series, we may consult [6, 17] implicitly.

Theorem 2.1. *For any $\alpha > 0$, the abscissa of convergence and absolute convergence of $Z(\alpha, s)$ is equal to one.*

Proof. Noting that

$$H_\alpha(x) := \sum_{n \le x} s_\alpha(n) = \lceil \alpha \lfloor x \rfloor \rceil,$$

we evaluate $\limsup_{x \to \infty} \log H_\alpha(x) / \log x = 1$. $\square$

For the present, we do not know to what extent $Z(\alpha, s)$ is continued analytically.

We end this section with an integral relation between Sturmian Dirichlet series and power series. See [6, Theorem 11] for its proof.

Theorem 2.2. *Let $\alpha > 0$. Then*

$$Z(\alpha, s) = \frac{1}{\Gamma(s)} \int_0^1 \left(\log \frac{1}{y} \right)^{s-1} \frac{\Xi(\alpha, y)}{y} \, dy,$$

if $\mathrm{Re}(s) > 1$.

3. Preliminaries.

The remainder term of our Dirichlet series obeys the next asymptotic behavior.

Lemma 3.1. *Let $\alpha > 0$ and $\mathrm{Re}(s) > 1$. Then*

$$\sum_{n=N+1}^{\infty} \frac{s_\alpha(n)}{n^s} = \frac{\alpha}{(s-1)N^{s-1}} + O(N^{-s}).$$

Proof. From identities [17, Theorem 1.3]

$$\sum_{n=1}^{N} \frac{s_\alpha(n)}{n^s} = \frac{H_\alpha(N)}{N^s} + s \int_1^N \frac{H_\alpha(x)}{x^{s+1}} \, dx, \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{s_\alpha(n)}{n^s} = s \int_1^{\infty} \frac{H_\alpha(x)}{x^{s+1}} \, dx, \tag{1}$$

one deduces that

$$\begin{aligned}\sum_{n=N+1}^{\infty} \frac{s_\alpha(n)}{n^s} &= -\frac{H_\alpha(N)}{N^s} + s\int_N^\infty \frac{H_\alpha(x)}{x^{s+1}}\,dx \\ &= -\frac{\alpha}{N^{s-1}} + O(N^{-s}) + s\int_N^\infty (\alpha x^{-s} + O(x^{-s-1}))\,dx \\ &= \frac{\alpha}{(s-1)N^{s-1}} + O(N^{-s}).\end{aligned}$$

□

From now on, we fix a real $\sigma > 1$, and regard $\nu_\sigma(x) := Z(x, \sigma)$ as a real function defined on $[0, \infty)$. For an integer $\alpha \geq 0$, we have $\nu_\sigma(\alpha) = \alpha\zeta(\sigma)$. Even if α is not an integer, one readily notes that

$$\nu_\sigma(\alpha) = \lfloor\alpha\rfloor\zeta(\sigma) + \nu_\sigma(\{\alpha\}),$$

where ζ is the Riemann zeta function. We, accordingly, may restrict the domain of ν_σ to $[0, 1]$ with no loss of generality.

Theorem 3.2. *For fixed $\sigma > 1$, the function $\nu_\sigma(x)$ is a strictly increasing function of x.*

Proof. If $\alpha_1 < \alpha_2$, then $H_{\alpha_1}(x) < H_{\alpha_2}(x)$ for all x sufficiently large. The result follows from (1):

$$\nu_\sigma(\alpha) = \sigma \int_1^\infty \frac{H_\alpha(x)}{x^{\sigma+1}} dx.$$

□

The differentiability of $\nu_\sigma(x)$ at $x = \alpha$ depends crucially on the Diophantine approximation of α. We need some Diophantine preliminaries. For any real t, we mean by $\|t\|$ the distance from t to the nearest integer:

$$\|t\| := \min_{n\in\mathbb{Z}} |t - n|.$$

An irrational number α is said to have *irrationality exponent* $\mu(\alpha)$ if

$$\mu(\alpha) := \sup\{\lambda : \liminf_{\substack{q\in\mathbb{N} \\ q\to\infty}} q^{\lambda-1}\|q\alpha\| = 0\}.$$

It is well known that $\mu(\alpha) \geq 2$ for every irrational α, and that actually $\mu(\alpha) = 2$ for almost every α (Khinchin's Theorem). In particular, every irrational algebraic number has irrationality exponent 2 (Roth's Theorem).

Lemma 3.3. *Let $\alpha > 0$ be irrational, and set*

$$\delta_N := \min\left\{\frac{\|\alpha n\|}{n} : 1 \le n \le N\right\}. \tag{2}$$

We have the following, for any real $\lambda \ge 2$.

(a) *If $2 \le \mu(\alpha) < \lambda$, then there exists a natural number $N_0 \in \mathbb{N}$ such that*

$$\bigcup_{n=N_0}^{\infty} (n^{-\lambda}, \delta_n) = (0, \delta_{N_0}),$$

i.e., two consecutive intervals $(n^{-\lambda}, \delta_n)$ and $((n+1)^{-\lambda}, \delta_{n+1})$ eventually overlap.

(b) *If $2 \le \lambda < \mu(\alpha)$, then*

$$Q := \left\{m_n : \delta_n = \frac{\|\alpha m_n\|}{m_n} < \frac{1}{2m_n^{\lambda}},\ n \in \mathbb{N}\right\}$$

is an infinite set, and every number in Q is the denominator of a convergent of the regular continued fraction for α.

Part (b) is not indispensable below, but is included for completeness.

Proof. (a) First, one observes that $n^{-\lambda} < \delta_n$ for all n sufficiently large. We claim that there exists a natural number $N_0 \in \mathbb{N}$ such that $\delta_n > (n-1)^{-\lambda}$ for all $n \ge N_0$. Suppose, on the contrary, that $L := \{n \in \mathbb{N} : \delta_n \le (n-1)^{-\lambda}\}$ is an infinite set. Then the set $\{m_n : \delta_n = \|\alpha m_n\|/m_n,\ n \in L\}$ is also infinite; otherwise, α should be a rational number. Let us pick κ so that $\mu(\alpha) < \kappa < \lambda$. We derive, for any $n \in L$,

$$m_n^{\kappa}\frac{\|\alpha m_n\|}{m_n} = m_n^{\kappa}\delta_n \le \frac{m_n^{\kappa}}{(n-1)^{\lambda}} \le \frac{n^{\kappa}}{(n-1)^{\lambda}} = \frac{1}{n^{\lambda-\kappa}}\frac{1}{(1-n^{-1})^{\lambda}} \to 0,$$

as $n \to \infty$. We obtain $\mu(\alpha) \ge \kappa$, a contradiction.

(b) For any $q \in Q$, there is $p \in \mathbb{N}$ such that

$$\left|\alpha - \frac{p}{q}\right| = \frac{\|\alpha q\|}{q} < \frac{1}{2q^{\lambda}} \le \frac{1}{2q^2}.$$

Since α is irrational, p and q are relatively prime, and therefore p/q is a convergent of α by Legendre's theorem. ∎

The last preliminary needed in the main results is to recall combinatorics and analysis on mechanical words.

For a rational $\alpha \in [0,1]$, it is obvious that s_α is a purely periodic word in $\{0,1\}^{\mathbb{N}}$. More precisely, if $\alpha = p/q \in [0,1]$ with $\gcd(p,q) = 1$, then the shortest period word of s_α has length q, and is of the form

$$s_\alpha = (1z_{p,q}0)^\omega,$$

for some word $z_{p,q}$ of length $q-2$. The word $z_{p,q}$ is known to be a palindrome, i.e., a word coinciding with its reversal [16, Corollary 2.2.9]. If $q = 2$, then $z_{p,q}$ is the empty word. In addition, if either $\alpha = 0/1$ or $1/1$, then we need the following conventions:

$$1z_{0,1}0 = 0, \quad 1z_{1,1}0 = 1.$$

A quick look shows that $\lim_{\alpha\to 0+} s_\alpha = 10^\omega = 1000\cdots$. In this limit, the topology on $\{0,1\}^{\mathbb{N}}$ is, as usual, the product topology of the discrete one on $\{0,1\}$. The next lemma specifies the limits of s_α at positive rationals.

Lemma 3.4. *Let $p/q \in [0,1)$ be rational with* $\gcd(p,q) = 1$. *Then*

- $\lim_{\alpha\to(p/q)-} s_\alpha = s_{p/q} = (1z_{p,q}0)^\omega$,
- $\lim_{\alpha\to(p/q)+} s_\alpha = 1(z_{p,q}10)^\omega$.

Proof. See [13, Lemma 2.3]. □

4. Calculus on Sturmian Dirichlet series.

In this section we study properties of the function $\nu_\sigma : (0,\infty) \to \mathbb{R}$, its continuity, limit behavior and differentiability.

First, we look at its behavior on the set of rationals. The function $\nu_\sigma(x)$ is discontinuous at every rational as the next theorem says. For an infinite word $a_1a_2\cdots$ of numbers, we write $\nu_\sigma(a_1a_2\cdots) := \sum_{n=1}^\infty a_n n^{-\sigma}$. For instance, $\nu_\sigma(p/q) = \nu_\sigma((1z_{p,q}0)^\omega)$ when $\gcd(p,q) = 1$.

Theorem 4.1. *Let $\sigma > 1$ be fixed and let $p/q \in [0,1)$ be rational with* $\gcd(p,q) = 1$. *The following hold.*

(a) $\lim_{\alpha\to(p/q)+} \nu_\sigma(\alpha) = \nu_\sigma(1(z_{p,q}10)^\omega)$.
(b) *At every rational, $\nu_\sigma(x)$ is left-continuous but not right-continuous.*

Proof. (a) If $p/q = 0/1$, then $z_{p,q}10$ should read $z_{p,q}10 = 0$. Lemma 3.4 guarantees that if $\alpha - p/q > 0$ is sufficiently small, then both s_α and $1(z_{p,q}10)^\omega$ have a common prefix of length N. By Lemma 3.1, one finds that

$$|\nu_\sigma(\alpha) - \nu_\sigma(1(z_{p,q}10)^\omega)| < C \cdot N^{-(\sigma-1)} + O(N^{-\sigma}),$$

where the constant C is independent of N. Since N is arbitrary, we are done.

(b) Since $\nu_\sigma(0) = 0$ and $\nu_\sigma(0+) = 1$, the function $\nu_\sigma(x)$ is not right-continuous at $x = 0$. Suppose $0 < p/q < 1$, and define

$$\delta_N := \min\left\{ \frac{\{(p/q)\cdot n\}}{n} : 1 \le n \le N,\ n \not\equiv 0 \pmod{q} \right\}. \tag{3}$$

Then $0 \le p/q - \alpha < \delta_N$ implies that both s_α and $(1z_{p,q}0)^\omega$ have a common prefix of length N. Therefore, we have

$$\nu_\sigma(p/q) - \nu_\sigma(\alpha) = \frac{p/q - \alpha}{(\sigma-1)N^{\sigma-1}} + O(N^{-\sigma}) \to 0 \text{ as } N \to \infty.$$

To prove that $\nu_\sigma(x)$ is not right-continuous at $x = p/q$, we observe that

$$\begin{aligned}\nu_\sigma(1(z_{p,q}10)^\omega) - \nu_\sigma((1z_{p,q}0)^\omega) &= \frac{1}{q^\sigma} - \frac{1}{(q+1)^\sigma} + \frac{1}{(2q)^\sigma} - \frac{1}{(2q+1)^\sigma} + \cdots \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{(nq)^\sigma} - \frac{1}{(nq+1)^\sigma} \right).\end{aligned}$$

The hypothesis $\sigma > 1$ and hence the absolute convergence of the series justifies the rearrangement of summation. Since each term in the last summation is positive, we have $\nu_\sigma(1(z_{p,q}10)^\omega) - \nu_\sigma(p/q) > 0$. $\square$

Though it is discontinuous, $\nu_\sigma(x)$ allows its left- and right-derivatives at each rational number. Recall that the Hurwitz zeta function $\zeta(s, a)$ is defined by

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \mathrm{Re}(s) > 1.$$

If $a = 1$, this reduces to the Riemann zeta function; $\zeta(s, 1) = \zeta(s)$.

Theorem 4.2. *Let $\sigma > 1$ be fixed and let $p/q \in (0, 1)$ be rational with* $\gcd(p, q) = 1$. *We have the following.*

(a) $\lim_{\alpha \to 0+} \frac{\nu_\sigma(\alpha)-1}{\alpha} = 0$.

(b) $\lim_{\alpha \to (p/q)-} \frac{\nu_\sigma(p/q)-\nu_\sigma(\alpha)}{p/q-\alpha} = 0$.

(c) $\lim_{\alpha \to (p/q)+} \frac{\nu_\sigma(\alpha)-\nu_\sigma(1(z_{p,q}10)^\omega)}{\alpha-p/q} = 0$.

Proof. (a) Let $(\alpha_n)_{n\geq 1}$ be a positive sequence converging to zero. We claim that $\lim_{n\to\infty} \frac{\nu_\sigma(\alpha_n)-1}{\alpha_n} = 0$. Since $\bigcup_{q=1}^{\infty}((2q)^{-1}, q^{-1}] = (0,1]$, one finds that α_n eventually belongs to some interval $((2q)^{-1}, q^{-1}]$, and further that $q \to \infty$ as $n \to \infty$. If $(2q)^{-1} < \alpha \leq q^{-1}$, then our claim can be inferred from

$$\begin{aligned}
0 < \frac{\nu_\sigma(\alpha)-1}{\alpha} < \frac{\nu_\sigma(q^{-1})-1}{(2q)^{-1}} &= 2q\left(\frac{1}{(q+1)^\sigma} + \frac{1}{(2q+1)^\sigma} + \cdots\right) \\
&= \frac{2q}{q^\sigma}\left(\frac{1}{(1+q^{-1})^\sigma} + \frac{1}{(2+q^{-1})^\sigma} + \cdots\right) \\
&= \frac{2}{q^{\sigma-1}}\zeta(\sigma, 1+q^{-1}) \to 0 \text{ as } q \to \infty.
\end{aligned}$$

The last convergence follows from $\sigma > 1$ and from $\lim_{q\to\infty} \zeta(\sigma, 1+q^{-1}) = \zeta(\sigma)$.

(b) Let $(\alpha_n)_{n\geq 1}$ be a sequence converging to p/q with $\alpha_n < p/q$. Define δ_N as in (3). If $n \not\equiv 0 \pmod q$, then $\{(p/q)\cdot n\} = r/q$ for some integer $0 < r < q$, from which it follows that

$$\delta_N \geq \frac{1}{qN} \quad \text{and} \quad \delta_{N+1} \geq \frac{1}{q(N+1)} \geq \frac{1}{2qN}. \tag{4}$$

Hence, we have

$$\bigcup_{N=1}^{\infty} ((2qN)^{-1}, \delta_N] = (0, p/q].$$

Now the sequence $p/q - \alpha_n$ eventually lies in an interval $((2qN)^{-1}, \delta_N]$ for some $N \geq 1$ with $N \to \infty$ as $n \to \infty$. If $(2qN)^{-1} < p/q - \alpha < \delta_N$, then both $s_{p/q}$ and s_α have a common prefix of length N. Thus, Lemma 3.1 shows that

$$\begin{aligned}
\frac{\nu_\sigma(p/q)-\nu_\sigma(\alpha)}{p/q-\alpha} &= \frac{1}{p/q-\alpha}\left(\sum_{n=N+1}^{\infty}\frac{s_{p/q}(n)}{n^\sigma} - \sum_{n=N+1}^{\infty}\frac{s_\alpha(n)}{n^\sigma}\right) \\
&= \frac{1}{p/q-\alpha}\left(\frac{p/q-\alpha}{(\sigma-1)N^{\sigma-1}} + O(N^{-\sigma})\right) \\
&< \frac{1}{(\sigma-1)N^{\sigma-1}} + O(\frac{2qN}{N^\sigma}) \to 0 \text{ as } N \to \infty.
\end{aligned}$$

(c) We also appeal to the reasoning of (a) and (b), but, in this case, let us define δ_N by

$$\delta_N := \min \left\{ \frac{1 - \{(p/q) \cdot n\}}{n} : 1 \leq n \leq N,\ n \not\equiv 0 \pmod q \right\}.$$

Then the two inequalities in (4) hold, too, and

$$\bigcup_{N=1}^{\infty} ((2qN)^{-1}, \delta_N] = (0, 1 - p/q].$$

Let α be such that $(2qN)^{-1} < \alpha - p/q < \delta_N$. Then both s_α and $1(z_{p,q}10)^\omega$ have a common prefix of length N. If $H'_\alpha(N)$ denotes the number of 1's in the first N letters of $1(z_{p,q}10)^\omega$, then $H'_\alpha(N) - H_\alpha(N)$ is at most one. So, we can suitably apply Lemma 3.1 to $\nu_\sigma(1(z_{p,q}10)^\omega)$. One, therefore, deduces that

$$\begin{aligned}
\frac{\nu_\sigma(\alpha) - \nu_\sigma(1(z_{p,q}10)^\omega)}{\alpha - p/q} &= \frac{1}{\alpha - p/q} \left(\frac{\alpha - p/q}{(\sigma - 1)N^{\sigma-1}} + O(N^{-\sigma}) \right) \\
&< \frac{1}{(\sigma - 1)N^{\sigma-1}} + O(\frac{2qN}{N^\sigma}) \to 0 \text{ as } N \to \infty.
\end{aligned}$$

□

We now turn to how the function $\nu_\sigma(x)$ behaves on the set of irrational numbers, which is the main result of this paper. The proof of the next theorem implicitly uses the fact that $l = \lim_{x \to x_0} f(x)$ if and only if $l = \lim_{n\to\infty} f(x_n)$ for every sequence $(x_n)_{n\geq1}$ with $x_n \neq x_0$ and $x_n \to x_0$.

Theorem 4.3. *Let $\alpha > 0$ be irrational.*

(a) *For fixed $\sigma > 1$, the function $\nu_\sigma(x)$ is continuous at every irrational.*
(b) *If $2 \leq \mu(\alpha) < \sigma$, then $\nu_\sigma(x)$ is differentiable at $x = \alpha$ and $\nu'_\sigma(\alpha) = 0$.*

Proof. Define δ_N as in (2). If positive α' satisfies $|\alpha' - \alpha| < \delta_N$, then the mechanical words $s_{\alpha'}$ and s_α have a common prefix of length N, because there is no integer lattice point other than the origin in the closed region circumscribed by the lines $y = \alpha x$, $y = \alpha' x$, and $x = N$. By Lemma 3.1, we have

$$|\nu_\sigma(\alpha') - \nu_\sigma(\alpha)| = \left| \frac{\alpha' - \alpha}{(\sigma - 1)N^{\sigma-1}} + O(N^{-\sigma}) \right|.$$

Letting $N \to \infty$, this value can be arbitrarily small.

Choose an arbitrary positive sequence $(\alpha_k)_{k\geq 1}$ so that $\alpha_k \neq \alpha$ and $\alpha_k \to \alpha$. And fix λ with $\mu(\alpha) < \lambda < \sigma$. By Lemma 3.3, the sequence $|\alpha_k - \alpha|$ eventually belongs to the union $\bigcup_{n=N_0}^{\infty}(n^{-\lambda}, \delta_n) = (0, \delta_{N_0})$ for some N_0. Consequently, if k is large enough, then we have $n^{-\lambda} < |\alpha_k - \alpha| < \delta_n$ for some $n \geq N_0$, and moreover $n \to \infty$ as $k \to \infty$. One concludes that

$$\begin{aligned} \frac{|\nu_\sigma(\alpha_k) - \nu_\sigma(\alpha)|}{|\alpha_k - \alpha|} &= \left| \frac{1}{(\sigma-1)n^{\sigma-1}} + O\left(\frac{1}{|\alpha_k - \alpha| n^\sigma}\right) \right| \\ &< \frac{1}{(\sigma-1)n^{\sigma-1}} + \left| O\left(\frac{1}{n^{\sigma-\lambda}}\right) \right| \to 0 \text{ as } k \to \infty. \end{aligned}$$

□

Remark 4.4. (i) Together with Khinchin's theorem, Theorem 4.3 implies that ν_σ is a singular function whenever $\sigma > 2$. In particular, at every positive irrational algebraic number, the derivative ν'_σ vanishes.

(ii) From the other inequality $\sigma < \mu(\alpha)$, it does not necessarily follows that ν_σ is not differentiable (cf. [14]). Since ν_σ is increasing (Theorem 3.2), the Lebesgue differentiation theorem ([15, 4]) guarantees that ν_σ is differentiable almost everywhere. Meanwhile, if $1 < \sigma < 2$, then Khinchin's theorem shows that $\sigma < \mu(\alpha)$ holds for almost every α. On the other hand, the case where $2 \leq \sigma \leq \mu(\alpha)$ seems to necessitate more detailed estimate on the error term than the one given in Lemma 3.1. Figure 1 illustrates the graph of $\nu_2(x)$.

(iii) For $\sigma > 2$, we have not a single example of α for which $\nu_\sigma(x)$ is either not differentiable at $x = \alpha$, or differentiable at $x = \alpha$ but $\nu'_\sigma(\alpha) > 0$. Such number is, however, very rare in view of Jarník's theorem, which states that the Hausdorff dimension of the set $\{\alpha \in \mathbb{R} : \mu(\alpha) \geq \sigma\}$ is equal to $2/\sigma$.

A real function $f(x)$ is said to satisfy a *Hölder condition* of *order* η at $x = x_0$ if η is the supremum of all γ for which

$$|f(x) - f(x_0)| \leq C|x - x_0|^\gamma, \quad C : \text{constant} \tag{5}$$

holds on some open interval containing x_0. In our context, this local version of Hölder condition is appropriate to measure how flat ν_σ is wherever its derivative vanishes.

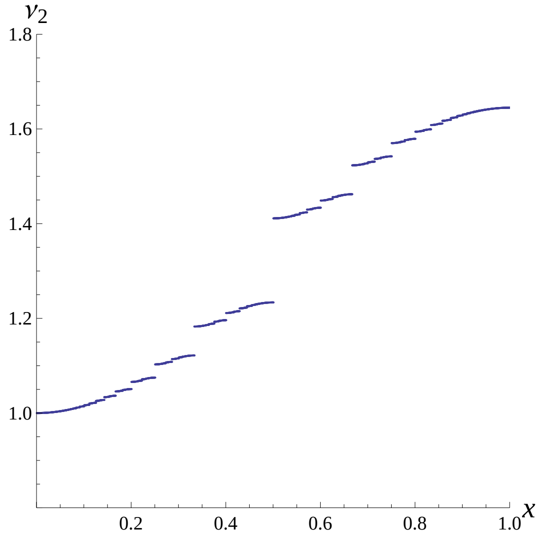

Figure 1: $\nu_2(x)$

Theorem 4.5. *Let $\alpha > 0$ be irrational and $\mu(\alpha) < \sigma$. Suppose that $\nu_\sigma(x)$ satisfies a Hölder condition of order η at $x = \alpha$. Then $\eta \geq \sigma/\mu(\alpha)$.*

Proof. We inherit the relevant argument from Theorem 4.3. Since $\nu'_\sigma(\alpha) = 0$, one has $\eta \geq 1$. Given λ with $\mu(\alpha) < \lambda < \sigma$, such sequence $(\alpha_k)_{k\geq 1}$ as in Theorem 4.3 fulfills

$$\begin{aligned}\frac{|\nu_\sigma(\alpha_k) - \nu_\sigma(\alpha)|}{|\alpha_k - \alpha|^\gamma} &= \left|\frac{1}{(\sigma-1)n^{\sigma-1}|\alpha_k - \alpha|^{\gamma-1}} + O\left(\frac{1}{|\alpha_k - \alpha|^\gamma n^\sigma}\right)\right| \\ &< \frac{1}{(\sigma-1)n^{\sigma-(\gamma-1)\lambda-1}} + \left|O\left(\frac{1}{n^{\sigma-\gamma\lambda}}\right)\right|.\end{aligned}$$

Now $\sigma - (\gamma-1)\lambda - 1 > 0$ (resp. $\sigma - \gamma\lambda > 0$) if and only if $\gamma < \frac{\sigma-1}{\lambda} + 1$ (resp. $\gamma < \frac{\sigma}{\lambda}$). Note $\frac{\sigma}{\lambda} < \frac{\sigma-1}{\lambda} + 1$. Letting $\lambda \searrow \mu(\alpha)$, we obtain the lower bound for order. $\square$

Remark 4.6. From the above theorem it is probable that, the bigger $\sigma/\mu(\alpha)$ is, the more flat $\nu_\sigma(x)$ is at $x = \alpha$. A more precise statement is possible by the upper bound of order η.

Acknowledgments.

To the anonymous referee, the author is gratefully indebted for detailed comments, from which better readability, citation of appropriate references, and correction of some mistakes in the first draft were all possible. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2013R1A1A2007508).

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