Discrete measures with dense jumps induced by Sturmian Dirichlet series

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Abstract

Let $(s_\alpha(n))_{n\geq 1}$ be the lexicographically greatest Sturmian word of slope $\alpha > 0$. For a fixed $\sigma > 1$, we consider Dirichlet series of the form $\nu_\sigma(\alpha) := \sum_{n=1}^{\infty} s_\alpha(n) n^{-\sigma}$. This paper studies the singular properties of the real function ν_σ, and the Lebesgue-Stieltjes measure whose distribution is given by ν_σ.

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1 Introduction

Throughout the paper, $\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes the set of positive (resp. non-negative) integers. We mean by $\lfloor\cdot\rfloor$ (resp. $\lceil\cdot\rceil$) the floor (resp. ceiling) function, and by A^* the set of finite words over the alphabet A, i.e., the free monoid generated by A.

For $\alpha \geq 0$, an arithmetic function $s_\alpha : \mathbb{N} \to \mathbb{N}_0$ is defined by

$$s_\alpha(n) := \lceil \alpha n \rceil - \lceil \alpha(n-1) \rceil.$$

Then $s_\alpha := s_\alpha(1)s_\alpha(2)\cdots$ is an infinite word over the alphabet $\{\lceil\alpha\rceil - 1, \lceil\alpha\rceil\}$. Now we set, for a fixed $\sigma > 1$,

$$\nu_\sigma(\alpha) := \sum_{n=1}^{\infty} \frac{s_\alpha(n)}{n^\sigma}, \tag{1}$$

i.e., Dirichlet series whose coefficients are given by s_α. From now on, we assume $\sigma > 1$ unless otherwise stated explicitly. This real function $\nu_\sigma : [0,\infty) \to \mathbb{R}$ was firstly considered in [3], and shown to be continuous at every irrational, whereas left-continuous but not right-continuous at every rational. Furthermore, ν_σ turned out to be singular. In other words, $\nu'_\sigma(\alpha) = 0$ for almost every α in the Lebesgue measure sense.

The present paper continues the study of its singularity in some depth. Since ν_σ is left-continuous everywhere, we can associate a Lebesgue-Stieltjes measure with it. Then this measure is singular with respect to the Lebesgue measure. We will prove that the measure is, actually, discrete or a countable summation of the Dirac measures, and moreover that the point masses are densely distributed. It is worthwhile to mention here that ν_σ is reminiscent of the function investigated in [2], which is induced by 'Sturmian power series'. They have a similar property in common from a measure-theoretical point of view. Many techniques in Sturmian power series also works in our context, but with careful modifications.

2 Preliminaries.

Analysis of ν_σ crucially relies upon the combinatorial properties of s_α. This preliminary section begins with combinatorics on words. Lothaire's book [4] is a standard reference.

Let $\alpha \geq 0$ and $\rho \in [0,1]$. We define, for $n \in \mathbb{N}_0$,

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$

$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

Then $s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots$ (resp. $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$), termed a *lower* (resp. *upper*) *mechanical word* with *slope* α and *intercept* ρ [5], is an infinite word over the alphabet $\{\lceil \alpha \rceil - 1, \lceil \alpha \rceil\}$. One recognizes that s_α is nothing but the upper mechanical word $s'_{\alpha,0}$. It readily follows that if α is an integer then $s_{\alpha,\rho} = s'_{\alpha,\rho} = \alpha^\omega := \alpha\alpha\cdots$ for any ρ. With this exception, both $\lceil \alpha \rceil - 1$ and $\lceil \alpha \rceil$ appear infinitely often in $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$. *Sturmian words* are mechanical words of irrational slopes, but they are sometimes used for general mechanical words in the literature.

Let $\alpha \geq 0$ be irrational and $\rho = 0$. We write $a := \lceil \alpha \rceil - 1$ and $b := \lceil \alpha \rceil$. Then both $s_{\alpha,0}$ and $s'_{\alpha,0}$ have a common infinite suffix $c_\alpha \in \{a,b\}^{\mathbb{N}}$, called

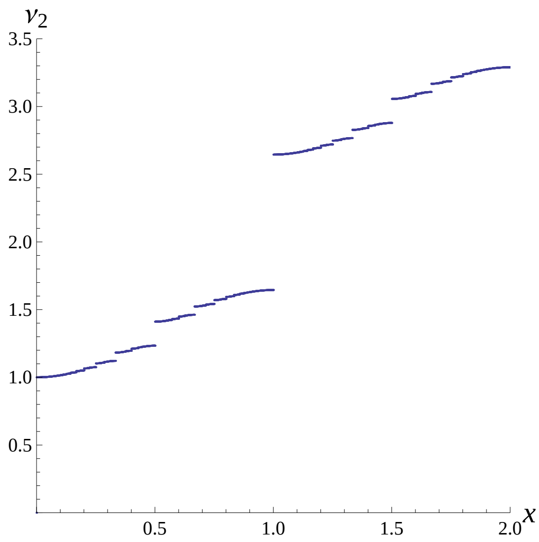

Figure 1: $\nu_2(x)$

the *characteristic word* of slope α:

$$s_{\alpha,0} = ac_\alpha, \quad s'_{\alpha,0} = bc_\alpha.$$

On the other hand, if $\alpha = p/q \geq 0$ is not an integer with $\gcd(p,q) = 1$, then obviously $s_{\alpha,0}$ and $s'_{\alpha,0}$ are purely periodic, and their shortest periodic words have a factor $z_{p,q} \in \{a,b\}^*$ of length $q-2$ in common:

$$s_{\alpha,0} = (az_{p,q}b)^\omega, \quad s'_{\alpha,0} = (bz_{p,q}a)^\omega.$$

The word $z_{p,q}$ is called the *central word* of slope p/q, which is known to be a palindrome — a word coinciding with its reversal.

The next proposition, demonstrated in Figure 1, is a summary from [3], which motivated the current work and will be pivotally used below. For an infinite word $a_1a_2\cdots \in \mathbb{N}_0^{\mathbb{N}}$, let us also write $\nu_\sigma(a_1a_2\cdots) := \sum_{n=1}^\infty a_n n^{-\sigma}$. For example, $\nu_\sigma(p/q) = \nu_\sigma((bz_{p,q}a)^\omega)$ when $\gcd(p,q) = 1$ and $b = a+1 = \lceil \alpha \rceil$.

Proposition 2.1. *Let $\sigma > 1$, and ν_σ be the function from $[0,\infty)$ into $\mathbb{R}$ defined in (1).*

(a) *The function ν_σ is strictly increasing.*

(b) *At every positive rational, ν_σ is left-continuous but not right-continuous.*

(c) *Let $p/q \geq 0$ be a rational with* $\gcd(p,q) = 1$*. Then the right-limit of ν_σ at p/q is given by*

$$\lim_{\alpha \to (p/q)+} \nu_\sigma(\alpha) = \nu_\sigma(b(z_{p,q}ba)^\omega)$$

while $\nu_\sigma(p/q) = \nu_\sigma((bz_{p,q}a)^\omega)$. In particular, if $q = 1$, then $b(z_{p,q}ba)^\omega$ (resp. $(bz_{p,q}a)^\omega$) should read ba^ω with $b = a+1 = p+1$ (resp. b^ω with $b = p$).

(d) *At every positive irrational, ν_σ is continuous, and moreover $\nu'_\sigma = 0$ almost everywhere.*

3 Integrations.

Suppose that a real or complex function f is bounded on an interval I. A classical theorem on integration states that f is Riemann-integrable on I if and only if f is continuous almost everywhere on I.

Since ν_σ is continuous almost everywhere, it allows the Riemann integration. This section integrates ν_σ over each interval $[a,b]$ in a closed form. As before, $b = a+1 = \lceil \alpha \rceil$. The integral sign in this section refers to the Riemann integral.

The hidden symmetry of the graph of ν_σ is revealed in the next lemma. It originally comes from the symmetry of mechanical words. Let E be the unique monoid automorphism on $\{a,b\}^*$ such that $E(a) = b$ and $E(b) = a$. This map naturally extends to $\{a,b\}^{\mathbb{N}}$.

Lemma 3.1. (a) *If $\alpha \geq 0$ is irrational, then $E(c_\alpha) = c_{a+b-\alpha}$.*

(b) *If $p/q \geq 0$ is rational with* $\gcd(p,q) = 1$*, then $E(z_{p,q}) = z_{(a+b)q-p,q}$.*

Proof. This is a restatement of [4, Lemma 2.2.17] in terms of $\{a,b\}$ rather than $\{0,1\}$. $\square$

On each interval $[a,b]$, ν_σ has the following type of symmetry, which we state in terms of the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and the Hurwitz zeta function $\zeta(s,h) := \sum_{n=0}^{\infty}(n+h)^{-s}$. For all number theory stuff in this article, the book, e.g., by Apostol [1] is enough.

Theorem 3.2. *Let $\alpha \geq 0$. Then*

$$\nu_\sigma(\alpha)+\nu_\sigma(a+b-\alpha) = \begin{cases} (a+b)\zeta(\sigma)+1, & \text{if } \alpha \text{ is irrational,} \\ (a+b-q^{-\sigma})\zeta(\sigma)+q^{-\sigma}\zeta(\sigma,q^{-1}), & \text{if } \alpha = \frac{p}{q} \text{ with } \gcd(p,q)=1. \end{cases}$$

Proof. If α is irrational, then

$$\nu_\sigma(\alpha)+\nu_\sigma(a+b-\alpha) = \nu_\sigma(bc_\alpha)+\nu_\sigma(bc_{a+b-\alpha}) = (a+b)\zeta(\sigma)+1.$$

Let $\alpha = p/q$ with $\gcd(p,q)=1$. One observes that

$$\begin{aligned} \nu_\sigma(\alpha)+\nu_\sigma(a+b-\alpha) &= \nu_\sigma((bz_{p,q}a)^\omega)+\nu_\sigma((bz_{(a+b)q-p,q}a)^\omega) \\ &= (a+b)\zeta(\sigma)+\left(\frac{1}{1^\sigma}+\frac{1}{(q+1)^\sigma}+\frac{1}{(2q+1)^\sigma}+\cdots\right) \\ &\quad -\left(\frac{1}{q^\sigma}+\frac{1}{(2q)^\sigma}+\cdots\right) \\ &= (a+b)\zeta(\sigma)+\frac{1}{q^\sigma}\zeta(\sigma,q^{-1})-\frac{1}{q^\sigma}\zeta(\sigma). \end{aligned}$$

Note that the absolute convergence of the series, which follows from the hypothesis $\sigma > 1$, justifies the rearrangement of summation. □

In the sense of Riemann integration, the set of rational points is negligible for bounded functions. The symmetry on the set of irrational points enables us to integrate ν_σ over $[a,b]$ in a closed form formula.

Corollary 3.2.1. *Let $b = a+1$ be a positive integer. Then ν_σ is Riemann-integrable on $[a,b]$ and*

$$\int_a^b \nu_\sigma(x)dx = \frac{a+b}{2}\zeta(\sigma)+\frac{1}{2}.$$

Proof. Recalling that $\nu_\sigma(\alpha)+\nu_\sigma(a+b-\alpha) = (a+b)\zeta(\sigma)+1$ for alomost every $\alpha \in [a,b]$, we have

$$2\int_a^b \nu_\sigma(x)dx = \int_a^b \nu_\sigma(x)dx + \int_a^b \nu_\sigma(a+b-x)dx = (a+b)\zeta(\sigma)+1.$$

□

4 Lebesgue-Stieltjes measures.

Let us denote by μ_σ the Lebesgue-Stieltjes measure associated with the function ν_σ. By Proposition 2.1, the measure μ_σ is singular with respect to the Lebesgue measure. According to the Lebesgue decomposition theorem, a singular measure is a singular continuous measure (e.g., the Cantor measure) plus a discrete measure (e.g., the Dirac measure). We will prove that μ_σ has no singular continuous part.

Lemma 4.1. *For any rational $p/q \geq 0$ with $\gcd(p,q) = 1$,*

$$\nu_\sigma((p/q)+) - \nu_\sigma(p/q) = q^{-\sigma}(\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})). \tag{2}$$

Proof. We find that

$$\begin{aligned}
\nu_\sigma((p/q)+) - \nu_\sigma(p/q) &= \nu_\sigma(b(z_{p,q}ba)^\omega) - \nu_\sigma((bz_{p,q}a)^\omega) \\
&= \nu_\sigma(bz_{p,q}b(az_{p,q}b)^\omega) - \nu_\sigma(bz_{p,q}a(bz_{p,q}a)^\omega) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{(nq)^\sigma} - \frac{1}{(nq+1)^\sigma} \right) \\
&= q^{-\sigma}(\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})).
\end{aligned}$$

□

Remark 4.2. (i) Note that the equality (2) is true even when p/q is an integer. In that case, we adhere to the convention introduced in Proposition 2.1.

(ii) The numerator p does not appear in the right-hand side of (2). This fact is realized in Figure 1. The graphs of ν_2 on $(0,1]$ and on $(1,2]$ are congruent.

The next theorem proves that μ_σ is a discrete measure whenever $\sigma > 2$, and further that its point masses are distributed over the whole nonnegative rational numbers. This is possible by establishing that the increases of the function ν_σ occur only at rational numbers and only by the amounts given in Lemma 4.1. Recall that if $\mathrm{Re}(s) > 2$, then

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)},$$

where φ is the Euler totient function.

Theorem 4.3. *Let $\sigma > 2$. Then μ_σ is a discrete measure. More precisely, for any $x \geq 0$, $\mu_\sigma([0,x))$ is equal to the sum of discontinuous jumps given in Lemma 4.1 at all rationals in $[0,x)$:*

$$\mu_\sigma([0,x)) = \sum_{\substack{0 \leq p/q < x \\ \gcd(p,q)=1}} \frac{\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})}{q^\sigma},$$

where the summation runs over all reduced rationals in $[0,x)$.

Proof. Since $\nu_\sigma((p/q)+) - \nu_\sigma(p/q)$ is independent of p, we may restrict the measure μ_σ to $[0,1)$, and it suffices to show that

$$\mu_\sigma = \sum_{\substack{0 \leq p/q < 1 \\ \gcd(p,q)=1}} \frac{\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})}{q^\sigma} \delta_{p/q}, \tag{3}$$

where δ_t is the Dirac measure. Plugging $\nu_\sigma(0+) = 1$ and $\nu_\sigma(1) = \zeta(\sigma)$, we know that

$$\mu_\sigma((0,1)) = \mu_\sigma([0,1)) - \mu_\sigma(\{0\}) = \zeta(\sigma) - 1.$$

Hence, Lemma 4.1 implies that (3) is equivalent to

$$\zeta(\sigma) - 1 = \sum_{\substack{0 < p/q < 1 \\ \gcd(p,q)=1}} \frac{\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})}{q^\sigma}.$$

One derives the right-hand side as follows:

$$\begin{aligned}
&\sum_{\substack{0 < p/q < 1 \\ \gcd(p,q)=1}} \frac{\zeta(\sigma) - \zeta(\sigma, 1+q^{-1})}{q^\sigma} \\
&\quad = \zeta(\sigma) \left(\sum_{q=1}^{\infty} \frac{\varphi(q)}{q^\sigma} - 1 \right) - \sum_{q=1}^{\infty} \frac{\varphi(q)}{q^\sigma} \zeta(\sigma, 1+q^{-1}) + \zeta(\sigma, 2) \\
&\quad = \zeta(\sigma - 1) - \zeta(\sigma) - \sum_{q=1}^{\infty} \frac{\varphi(q)}{q^\sigma} \zeta(\sigma, 1+q^{-1}) + \zeta(\sigma) - 1,
\end{aligned}$$

where

$$\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^\sigma}\zeta(\sigma,1+q^{-1}) = \sum_{q=1}^{\infty}\frac{\varphi(q)}{q^\sigma}\sum_{n=0}^{\infty}\frac{1}{(n+1+q^{-1})^\sigma} = \sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\frac{\varphi(q)}{(q(n+1)+1)^\sigma}$$
$$= \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\frac{\varphi(q)}{(qn+1)^\sigma} = \sum_{m=1}^{\infty}\sum_{q|m}\frac{\varphi(q)}{(m+1)^\sigma}$$
$$= \sum_{m=1}^{\infty}\frac{1}{(m+1)^\sigma}\sum_{q|m}\varphi(q) = \sum_{m=1}^{\infty}\frac{m}{(m+1)^\sigma}$$
$$= \sum_{m=1}^{\infty}\frac{1}{(m+1)^{\sigma-1}} - \sum_{m=1}^{\infty}\frac{1}{(m+1)^\sigma} = \zeta(\sigma-1)-\zeta(\sigma).$$

□

In terms of the function ν_σ, the previous theorem actually reveals point-set topology of the image of ν_σ. We let $\mathbb{R}_+$ denote the set of positive real numbers.

Theorem 4.4. *The closure of $\nu_\sigma(\mathbb{R}_+)$ is of Lebesgue measure zero, perfect, and nowhere dense.*

Proof. Theorem 4.3 proves that $[0,\infty)\setminus\overline{\nu_\sigma(\mathbb{R}_+)}$ has full Lebesgue measure.

Let $y \in \overline{\nu_\sigma(\mathbb{R}_+)}$. Then either $y = \nu_\sigma(bc_\alpha)$ for some irrational $\alpha > 0$, or $y = \nu_\sigma((bz_{p,q}a)^\omega)$ or $y = \nu_\sigma(b(z_{p,q}ba)^\omega)$ for some rational $\alpha = p/q > 0$. If $y = \nu_\sigma(bc_\alpha)$ or if $y = \nu_\sigma((bz_{p,q}a)^\omega)$, then we choose an arbitrary strictly increasing sequence $(\alpha_n)_{n\geq 1}$ converging to α. Now every neighborhood of y contains some $\nu_\sigma(\alpha_n)$. If $y = \nu_\sigma(b(z_{p,q}ba)^\omega)$, then a strictly decreasing sequence $(\alpha_n)_{n\geq 1}$ converging to α plays a similar role.

Suppose that $\overline{\nu_\sigma(\mathbb{R}_+)}$ has nonempty interior, and hence contains a closed interval $[y_1,y_2]$. We may assume that $y_1 = \nu_\sigma(bc_{\alpha_1})$ and $y_2 = \nu_\sigma(bc_{\alpha_2})$ for some irrational α_1 and α_2 with $\alpha_1 < \alpha_2$. Let us pick a rational $\alpha = p/q$ in (α_1,α_2). Then an open interval $(\nu_\sigma((bz_{p,q}a)^\omega), \nu_\sigma(b(z_{p,q}ba)^\omega))$ cannot be contained in $\overline{\nu_\sigma(\mathbb{R}_+)}$, which is a contradiction. □

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