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FOURIER SERIES OF A DEVIL'S STAIRCASE

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Abstract. Given $\beta > 1$, we consider real numbers whose β-expansions are Sturmian words. When the slope of Sturmian words varies, their behaviors have been well studied from analytical point of view. The regularity enables us to find the Fourier series expansion, while the singularity at rational slopes yields a new kind of trigonometric series representing π.

1. Introduction

For a nonempty subset A of $\mathbb{R}$, we denote by χ_A the characteristic function of A, i.e., $\chi_A(x) = 1$ whenever $x \in A$, and $\chi_A(x) = 0$ otherwise. Let us define a real function f by $f(x) := \chi_{(0,\pi)}(x) - \chi_{(-\pi,0)}(x)$ for $x \in [-\pi, \pi]$ with $f(x + 2\pi) = f(x)$ for every $x \in \mathbb{R}$. Then the periodic function f allows the Fourier series expansion

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x, \quad x \in \mathbb{R}.$$

Plugging $x = \pi/2$ into this formula, one obtains the well-known Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots . \tag{1}$$

A devil's staircase (or a singular function) is, by definition, a real function whose derivative vanishes almost everywhere. In the present paper, we construct a devil's staircase combining Sturmian words and β-expansions. More precisely, this function maps the slopes of Sturmian words to real numbers whose β-expansions are Sturmian words of the corresponding slopes. And then, its Fourier series will be computed.

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While Fourier analysis on devil's staircases seems to be not much studied in the literature, our function narrowly escapes the barrier against having its Fourier series. As a byproduct, the Fourier series gives us a new kind of trigonometric series representing π. A typical identity yielded by this Fourier series is, for example, the following:

$$\frac{\pi}{10} = \sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1}}{2k-1} \left(\sum_{d|2k-1} \frac{d}{2^d} \right) \right]. \tag{2}$$

2. Preliminaries

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ be the floor and ceiling functions respectively, and $\mathbb{N}$ the set of nonnegative integers. For a finite alphabet A, we mean by A^*, the free monoid generated by A. Given real $\alpha, \rho \in [0,1]$, we define two arithmetic functions $s_{\alpha,\rho}, s'_{\alpha,\rho} : \mathbb{N} \to \{0,1\}$ by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$
$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil,$$

to obtain infinite words

$$s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots \quad \text{and} \quad s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots.$$

Here, the word $s_{\alpha,\rho}$ (resp. $s'_{\alpha,\rho}$) is called a *lower* (resp. *upper*) *mechanical word* with *slope* α and *intercept* ρ. The combinatorics on these words has been hugely accumulated since Morse and Hedlund [7] introduced them.

When the slope α is irrational, both $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are termed *Sturmian words*. If, in addition, $\rho = 0$, then two Sturmian words $s_{\alpha,0}$ and $s'_{\alpha,0}$ have a common infinite suffix c_α as follows:

$$s_{\alpha,0} = 0c_\alpha, \quad s'_{\alpha,0} = 1c_\alpha,$$

where the common word c_α is called the *characteristic word* of slope α.

On the other hand, the rational slope α forces both $s_{\alpha,0}$ and $s'_{\alpha,0}$ to be purely periodic. If, say, $\alpha = p/q \in [0,1]$ with $\gcd(p,q) = 1$, then their shortest periodic words are of the form $0z_{p,q}1$ and $1z_{p,q}0$, respectively for some common finite word $z_{p,q} \in \{0,1\}^*$:

$$s_{\alpha,0} = (0z_{p,q}1)^\infty, \quad s'_{\alpha,0} = (1z_{p,q}0)^\infty.$$

For a finite word w, we mean by w^∞ the infinite words $www\cdots$. The word $z_{p,q}$, called the *central word*, is known to be a palindrome — a word coinciding with the reversal of itself, and its length is equal to $q-2$.

Set $E(0) := 1$ and $E(1) := 0$, and extend $E : \{0,1\}^* \to \{0,1\}^*$ so that E is a monoid homomorphism. We also write $\overline{u} := E(u)$ for a word $u \in \{0,1\}^*$. The central words possess the following type of symmetry.

Proposition 2.1. *Let $p/q \in [0,1]$ be a rational number with $\gcd(p,q) = 1$. Then*

$$E(z_{p,q}) = \overline{z_{p,q}} = z_{q-p,q}.$$

Proof. See [6]. □

For a fixed real $\alpha \in [0,1]$, $s'_{\alpha,0}$ is lexicographically greater than, and $s_{\alpha,0}$ smaller than any other $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ [1]. The readers who want to pursue further on mechanical words are recommended to begin with [6].

Since the introduction by Rényi [9] and Parry [8], β-expansions have been enjoying much interests and studies from diverse fields of mathematics. Among others, a connection between β-expansions and Sturmian words turned out to be a fertile field. We consider the case where the β-expansions are Sturmian words as follows.

Let us define a two-variable function $\Xi : [0,1] \times (1,\infty) \to \mathbb{R}$ by

$$\Xi(\alpha,\beta) := (s'_{\alpha,0})_\beta := \sum_{n=0}^{\infty} \frac{s'_{\alpha,0}(n)}{\beta^{n+1}}.$$

Calculus and arithmetic on the function Ξ were investigated in [4], while the level curve $\Xi(\alpha,\beta) = 1$ curiously and naturally emerged when generalizing classical baker's transformations [3]. In contrast, this paper fixes $\beta > 1$, and considers a one-variable function

$$f_\beta(x) := \Xi(x,\beta),$$

which is also a main theme of [5]. We summarize the main properties of f_β from there.

Proposition 2.2. *Let $\beta > 1$ and $f_\beta : (0,1] \to \mathbb{R}$ be defined as above. Then the function f_β satisfies the following.*

(a) *$f_\beta(x)$ is strictly increasing.*
(b) *$f_\beta(x)$ is continuous at $x = \alpha$ if and only if α is irrational.*
(c) *At a rational α, $f_\beta(x)$ is left-continuous but not right-continuous.*
(d) *If $\alpha = p/q \in [0,1]$ with $\gcd(p,q) = 1$, then $f_\beta(\alpha) = ((1z_{p,q}0)^\infty)_\beta$ and the right limit of $f_\beta(x)$ at $x = \alpha$ is given by*

$$f_\beta(\alpha+) := \lim_{x\to\alpha+} f_\beta(x) = (1(z_{p,q}10)^\infty)_\beta.$$

The case of $\alpha = 0$ is understood as

$$f_\beta(0+) = (1(z_{0,1}10)^\infty)_\beta := (1(0^\infty))_\beta = 1/\beta.$$

(e) *$f_\beta(x)$ is Riemann-integrable on $[0,1]$ and*

$$\int_0^1 f_\beta(x)\,dx = \frac{2\beta-1}{2\beta(\beta-1)}.$$

In the next section, we compute the Fourier series expansion of f_β. The previous proposition tells us that f_β behaves well enough that we apply Jordan's test for convergence of its Fourier series. This enables us to derive a new kind of trigonometric series representing π.

Let a function $f:\mathbb{R}\to\mathbb{C}$ with period 1 be in $L^1[0,1]$, and set, for a positive integer N,

$$s_N f(x) := \frac{a_0}{2} + \sum_{m=1}^{N}\left(a_m \cos 2m\pi x + b_m \sin 2m\pi x\right),$$

where

$$a_m := 2\int_0^1 f(x)\cos 2m\pi x\,dx \quad \text{for } m \geq 0,$$

$$b_m := 2\int_0^1 f(x)\sin 2m\pi x\,dx \quad \text{for } m \geq 1.$$

We say that $s_\infty f(x) := \lim_{N\to\infty} s_N f(x)$ is the *Fourier series* of f.

Proposition 2.3 (Jordan's test)**.** *Let a function $f:\mathbb{R}\to\mathbb{C}$ with period 1 be in $L^1[0,1]$. If there exists $\delta>0$ such that f is of bounded variation on $[x-\delta, x+\delta]$, then the Fourier series of f satisfies*

$$s_\infty f(x) = \frac{f(x+)+f(x-)}{2}.$$

In particular, if f is continuous at x, then

$$s_\infty f(x) = f(x).$$

Proof. See Section 10.1 of [2]. □

Recall that any monotone function is of bounded variation on a closed interval.

3. Fourier series expansions

We compute a_m and b_m for $f_\beta(x)$. For typographical convenience' sake, we adopt, for $n\geq 1$,

$$s_\alpha(n) := \lceil \alpha n\rceil - \lceil \alpha(n-1)\rceil \quad \text{and} \quad s_\alpha := s_\alpha(1)s_\alpha(2)\cdots.$$

In other words, $s_\alpha = s'_{\alpha,0}$. For any real $t \in \mathbb{R}$, we mean by $\{t\}$ the fractional part of t, i.e., $t = \lfloor t \rfloor + \{t\}$. It is readily checked that any $t \in \mathbb{R}$ fulfills

$$\lceil t \rceil = t + \{-t\}.$$

Consequently, we find

$$s_\alpha(n) = \alpha + \{-\alpha n\} - \{-\alpha(n-1)\},$$

and hence

$$f_\beta(x) = \sum_{n=1}^{\infty} \frac{x + \{-nx\} - \{-(n-1)x\}}{\beta^n}.$$

Lemma 3.1. *Let m and n be positive integers. Then*

$$\int_0^1 \{-nx\} \cos 2m\pi x \, dx = 0$$

and

$$\int_0^1 \{-nx\} \sin 2m\pi x \, dx = \begin{cases} \frac{n}{2m\pi}, & \text{if } n \text{ divides } m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $0 \le k < n$ be an integer. If $x \in (\frac{k}{n}, \frac{k+1}{n}]$, then $\lfloor -nx \rfloor = -k-1$. One thus derives

$$\begin{aligned} \int_{k/n}^{(k+1)/n} \{-nx\} \cos 2m\pi x \, dx &= \int_{k/n}^{(k+1)/n} (-nx + k + 1) \cos 2m\pi x \, dx \\ &= -\frac{1}{2m\pi} \sin \frac{2mk\pi}{n} + \frac{n}{(2m\pi)^2} \left(\cos \frac{2mk\pi}{n} - \cos \frac{2m(k+1)\pi}{n} \right), \end{aligned}$$

which is followed by

$$\begin{aligned} \int_0^1 \{-nx\} \cos 2m\pi x \, dx &= \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \{-nx\} \cos 2m\pi x \, dx \\ &= -\frac{1}{2m\pi} \sum_{k=0}^{n-1} \sin \frac{2mk\pi}{n} = 0. \end{aligned}$$

A similar argument shows that

$$\begin{aligned} &\int_{k/n}^{(k+1)/n} \{-nx\} \sin 2m\pi x \, dx \\ &\qquad = \frac{1}{2m\pi} \cos \frac{2mk\pi}{n} + \frac{n}{(2m\pi)^2} \left(\sin \frac{2mk\pi}{n} - \sin \frac{2m(k+1)\pi}{n} \right) \end{aligned}$$

and in turn that

$$\int_0^1 \{-nx\} \sin 2m\pi x \, dx = \frac{1}{2m\pi} \sum_{k=0}^{n-1} \cos \frac{2mk\pi}{n} = \begin{cases} \frac{n}{2m\pi}, & \text{if } n \text{ divides } m, \\ 0, & \text{otherwise.} \end{cases}$$

□

Now we compute the Fourier series of f_β. Let

$$s_\infty f_\beta(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos 2m\pi x + b_m \sin 2m\pi x \right).$$

Theorem 3.2. *We have*

$$s_\infty f_\beta(x) = \frac{2\beta - 1}{2\beta(\beta - 1)} + \sum_{m=1}^{\infty} b_m \sin 2m\pi x,$$

where, for $m \geq 1$,

$$b_m = \frac{1}{m\pi} \left(-\frac{1}{\beta - 1} + (\beta - 1) \sum_{n|m} \frac{n}{\beta^{n+1}} \right).$$

Proof. It follows from Proposition 2.2 that

$$\frac{a_0}{2} = \int_0^1 f_\beta(x) \, dx = \frac{2\beta - 1}{2\beta(\beta - 1)}.$$

For $m \geq 1$, Lemma 3.1 proves that

$$\begin{aligned} a_m &= 2 \int_0^1 f_\beta(x) \cos 2m\pi x \, dx \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{\beta^n} \left(\int_0^1 x \cos 2m\pi x \, dx + \int_0^1 \{-nx\} \cos 2m\pi x \, dx \right. \\ &\qquad \left. - \int_0^1 \{-(n-1)x\} \cos 2m\pi x \, dx \right) = 0. \end{aligned}$$

Here, the second equality rests upon the fact $\beta > 1$.

As for b_m, we find

$$\begin{aligned}
b_m &= 2\int_0^1 f_\beta(x)\sin 2m\pi x\,dx \\
&= 2\sum_{n=1}^{\infty}\frac{1}{\beta^n}\biggl(\int_0^1 x\sin 2m\pi x\,dx + \int_0^1 \{-nx\}\sin 2m\pi x\,dx \\
&\qquad\qquad - \int_0^1 \{-(n-1)x\}\sin 2m\pi x\,dx\biggr) \\
&= \frac{1}{m\pi}\left(-\sum_{n=1}^{\infty}\frac{1}{\beta^n} + \sum_{n|m}\frac{n}{\beta^n} - \sum_{n|m}\frac{n}{\beta^{n+1}}\right) \\
&= \frac{1}{m\pi}\left(-\frac{1}{\beta-1} + (\beta-1)\sum_{n|m}\frac{n}{\beta^{n+1}}\right).
\end{aligned}$$

□

4. Evaluations at rationals

If $\alpha \in (0,1)$ is irrational, then Proposition 2.3 proves that

$$f_\beta(\alpha) = \frac{2\beta-1}{2\beta(\beta-1)} + \sum_{m=1}^{\infty} b_m \sin 2m\pi\alpha,$$

where b_m is given as in Theorem 3.2. In this section, we evaluate $s_\infty f_\beta(x)$ at rational points.

Theorem 4.1. *Let* $\alpha = p/q \in (0,1)$ *be a rational number with* $\gcd(p,q) = 1$. *Then*

$$\frac{f_\beta(1-\alpha) - f_\beta(\alpha)}{2}\cdot\pi = \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{\beta-1} - (\beta-1)\sum_{n|m}\frac{n}{\beta^{n+1}}\right)\sin 2m\pi\alpha.$$

Proof. Since $f_\beta(x)$ is left-continuous, Jordan's test together with Theorem 3.2 guarantees that

$$\frac{f_\beta(\alpha+) + f_\beta(\alpha)}{2} = \frac{2\beta-1}{2\beta(\beta-1)} + \sum_{m=1}^{\infty} b_m \sin 2m\pi\alpha,$$

where

$$b_m = \frac{1}{m\pi}\left(-\frac{1}{\beta-1} + (\beta-1)\sum_{n|m}\frac{n}{\beta^{n+1}}\right).$$

By Proposition 2.2, one has

$$f_\beta(\alpha) = ((1z_{p,q}0)^\infty)_\beta \quad \text{and} \quad f_\beta(\alpha+) = (1(z_{p,q}10)^\infty)_\beta.$$

From

$$\frac{2\beta-1}{2\beta(\beta-1)} = \frac{1}{2}\left(\frac{1}{\beta}+\frac{1}{\beta-1}\right) = \frac{1}{2}\left[(10^\infty)_\beta + (1^\infty)_\beta\right],$$

and from Proposition 2.1, it follows that

$$\begin{aligned}
\sum_{m=1}^{\infty} b_m \sin 2m\pi\alpha &= \frac{f_\beta(\alpha+)+f_\beta(\alpha)}{2} - \frac{2\beta-1}{2\beta(\beta-1)} \\
&= \frac{1}{2}\left[(1(z_{p,q}10)^\infty)_\beta + ((1z_{p,q}0)^\infty)_\beta - (10^\infty)_\beta - (1^\infty)_\beta\right] \\
&= \frac{1}{2}\left[((0z_{p,q}1)^\infty)_\beta - ((0\overline{z_{p,q}}1)^\infty)_\beta\right] \\
&= \frac{1}{2}\left[((1z_{p,q}0)^\infty)_\beta - ((1z_{q-p,q}0)^\infty)_\beta\right] \\
&= \frac{1}{2}[f_\beta(\alpha) - f_\beta(1-\alpha)].
\end{aligned}$$

Multiplying both sides by $-\pi$, we obtain the conclusion. □

If $\alpha = 1/2$, then Theorem 4.1 gives us a trivial equation. For some illustrating examples, let us assume $\beta = 2$ from now on, that is,

$$\frac{f_2(1-\alpha)-f_2(\alpha)}{2}\cdot\pi = \sum_{m=1}^{\infty}\frac{1}{m}\left(1-\sum_{n|m}\frac{n}{2^{n+1}}\right)\sin 2m\pi\alpha$$

for any rational $\alpha \in (0,1)$

Example 1. Let $\alpha = 1/4$. Since we have

$$f_2(3/4) - f_2(1/4) = ((1110)^\infty)_2 - ((1000)^\infty)_2 = ((0110)^\infty)_2 = 2/5,$$

Theorem 4.1 reads

$$\begin{aligned}\frac{\pi}{5} &= \sum_{m=1}^{\infty} \frac{1}{m}\left(1-\sum_{n|m}\frac{n}{2^{n+1}}\right)\sin\frac{m\pi}{2} \\ &= \sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{2k-1}\left(1-\sum_{n|2k-1}\frac{n}{2^{n+1}}\right)\right] = \frac{3}{4}-\frac{3}{16}+\frac{43}{320}-\frac{185}{1792}+\cdots.\end{aligned}$$

The above formula together with the Leibniz formula (1) provides us with the identity (2).

Set

$$p(N) := 5\cdot\sum_{k=1}^{N}\left[\frac{(-1)^{k-1}}{2k-1}\left(1-\sum_{n|2k-1}\frac{n}{2^{n+1}}\right)\right].$$

With the aid of computing software such as Mathematica, we have the following:

$$p(99) \approx 3.153992836, \quad p(100) \approx 3.135148615,$$
$$p(999) \approx 3.142889102, \quad p(1000) \approx 3.141013164.$$

For instance, a Mathematica code

```
l=Table[DivisorSum[k,#/2^(#+1)&],{k,1,2000,2}];
N[5Sum[(-1)^(k-1)/(2k-1)*(1-l[[k]]),{k,1,1000}],10]
```

brings forth the value 3.141013164.

Example 2. For $\alpha = 1/3$, we deduce

$$f_2(2/3) - f_2(1/3) = ((110)^{\infty})_2 - ((100)^{\infty})_2 = ((010)^{\infty})_2 = 2/7,$$

which is followed by

$$\frac{\pi}{7} = \sum_{m=1}^{\infty}\frac{1}{m}\left(1-\sum_{n|m}\frac{n}{2^{n+1}}\right)\sin\frac{2m\pi}{3} = \frac{\sqrt{3}}{2}\left(\frac{3}{4}-\frac{1}{4}+\frac{3}{32}-\frac{43}{320}+\cdots\right).$$

Let $q(N)$ be the sum of the first N nonzero terms of

$$7\cdot\sum_{m=1}^{\infty}\frac{1}{m}\left(1-\sum_{n|m}\frac{n}{2^{n+1}}\right)\sin\frac{2m\pi}{3}.$$

Computations show that

$$q(99) \approx 3.154535158, \quad q(100) \approx 3.124020840,$$
$$q(999) \approx 3.143321715, \quad q(1000) \approx 3.140288604.$$

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