Moments of discrete measures with dense jumps induced by β-expansions

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Abstract

Let $\beta > 1$. Through an appeal to β-expansions we define a strictly increasing and left-continuous function μ_β on $[0, 1]$. Then μ_β turns out to be a pure jump distribution. In other words, its associated Lebesgue-Stieltjes measure is discrete, i.e., a summation of point masses. The present note studies the moment of this discrete measure and its asymptotics.

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1. Introduction

A numeration is a systematic means of representing numbers by finite or infinite words. Considering two different numerations at once, we have so far obtained some important (counter-)examples for analysis. Just as the ternary expansions together with the binary expansions produce the Cantor function, Minkowski's $?(x)$ function is obtained via the (regular) continued fraction and the alternated binary expansions [20]. Both functions are quite exceptional because they are *singular functions*, i.e., their derivatives vanish almost everywhere in the Lebesgue measure sense. They are continuous monotone functions. But Minkowski's $?(x)$ function is, unlike the Cantor function, strictly increasing. So the singularity of Minkowski's function was even more curious when Denjoy [7] and later Salem [27] proved it. Another strictly increasing singular function was also realized by Riesz and Sz.-Nágy [26]. Recently the Riesz-Nágy function was generalized in [23], where it was

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shown that its differentiability at x rests upon the normality of x to base 2. On the other hand, the differentiability of $?(x)$ is tightly related to the continued fraction of x [22, 9]. A more bald connection between differentiability and Diophantine property was demonstrated by a singular function $\Delta(x)$ in [13]. Although $\Delta(x)$ is discontinuous at every rational $x > 0$, it was proved that $\Delta(x)$ is differentiable unless x is extremely well approximable by rationals. In particular, at any non-Liouville irrational $x > 0$, the function Δ is differentiable and $\Delta'(x) = 0$, whereas, at x whose continued fraction is such as $x = [0; 1, 2^2, 3^{3^3}, 4^{4^{4^4}}, \ldots]$, Δ is not differentiable. The function Δ has many properties in common with *saltus functions* considered in [12]. Very recently, Δ was generalized to a two-variable function $\Xi : [0, \infty) \times [1, \infty) \to \mathbb{R}$ in [16], which is discontinuous on a dense set but total differentiable almost everywhere. Via Ξ, we constructed a two-variable singular function, that is, a non-constant continuous function that is locally constant on a set of full measure. The function Ξ is our main topic, and hence will be explicitly defined below.

The moments of the above mentioned distributions have been well studied in diverse contexts. For the moments of the Cantor function, see a survey [8] and the bibliography therein. And Alkauskas has been studying the moments of Minkowski's $?(x)$ function in a series of papers [2, 3, 4]. The moments of the generalized Riesz-Nágy distribution were computed by Baek [5] in terms of a recurrence relation. For a fixed $\beta > 1$, the present paper considers a distribution $\mu_\beta(x) := \Xi(x, \beta)$ on the interval $[0, 1]$, and analyzes its moments

$$M_m := \int_0^1 x^m d\mu_\beta, \quad m = 0, 1, 2, \ldots.$$

The asymptotics of M_m are also explored as m tends to infinity.

We will note below that, with being discontinuous at every rational, the function $\mu_\beta(x)$ is singular. And we will prove that it is a pure jump distribution. In other words, the Lebesgue-Stieltjes measure induced by this distribution is discrete, and point masses are distributed over the whole rationals in the interval $[0, 1]$. The moments of discrete measures with dense jumps were also considered in [19, 28] to investigate the asymptotic behavior of the coefficients of orthogonal polynomials.

In Section 2, we define the distribution μ_β with some preliminaries. Section 3 derives a closed-form formula for the moments of μ_β, and with this formula in hand, their asymptotics are investigated in Section 4. Some concrete examples are also presented in the last section.

2. Definition and property of μ_β.

Let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ be the floor and ceiling functions respectively. We denote by $\mathbb{N}$ the set of nonnegative integers. For real $\alpha, \rho \in [0,1]$, two functions $s_{\alpha,\rho}, s'_{\alpha,\rho} : \mathbb{N} \to \mathbb{N}$ are defined by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$

$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil,$$

which yield infinite words $s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots$ and $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$. Note that $s_{\alpha,\rho}(n) = s'_{\alpha,\rho}(n)$ provided that neither $\alpha n + \rho$ nor $\alpha(n+1)+\rho$ is equal to an integer. Actually, if $\alpha n + \rho$ is an integer for $\alpha \neq 0, 1$, then

$$s_{\alpha,\rho}(n) = 0,\ s'_{\alpha,\rho}(n) = 1 \ \text{ and } \ s_{\alpha,\rho}(n-1) = 1,\ s'_{\alpha,\rho}(n-1) = 0.$$

The word $s_{\alpha,\rho}$ (resp. $s'_{\alpha,\rho}$) is termed a *lower* (resp. *upper*) *mechanical word* with *slope* α and *intercept* ρ. It readily follows that $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are binary or unary words over the alphabet $A := \{0,1\}$. For $\alpha = 0$ or 1, $s_{\alpha,\rho} = s'_{\alpha,\rho} = \alpha^\omega := \alpha\alpha\cdots$. If α is neither 0 nor 1, then both 0 and 1 appear in $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$. This type of words has been a flourishing topic in combinatorics on words. The interested readers are referred to [18].

For $\beta > 1$, a function $(\cdot)_\beta$ is defined to send each infinite word $a_0a_1\cdots \in A^{\mathbb{N}}$ to a real number $\sum_{i=0}^\infty a_i/\beta^{i+1}$. If the word $a_0a_1\cdots$ satisfies some lexicographical condition, then we say that $a_0a_1\cdots$ is the β-expansion of $\sum_{i=0}^\infty a_i/\beta^{i+1}$ [24]. Since Rényi [25] and Parry [24] did it, numerous mathematicians studied β-expansions from an ergodic point of view. Now we define the function $\Xi : [0,1]\times(1,\infty) \to \mathbb{R}$ by

$$\Xi(\alpha,\beta) := (s'_{\alpha,0})_\beta.$$

Though α is restricted here to the unit interval $[0,1]$, it can be extended to $[0,\infty)$ as in [16]. Note that $s'_{\alpha,0}$ fulfills Parry's lexicographical condition so that $s'_{\alpha,0}$ is a β-expansion of 1 for some $\beta > 1$ (cf. [15, Proposition 3.2]). It is also worthwhile to mention that the map Ξ was alternatively embodied, in [15], by β-transformations. On the other hand, via the level curve $\Xi(\alpha,\beta) = 1$ in [14], the author completely characterized generalized baker's transformations in a natural sense.

Instead of $(s'_{\alpha,0})_\beta$ in the definition of Ξ, we can also study the case of $(s_{\alpha,0})_\beta$ with little modification. But, for a fixed $\rho \neq 0$, it is difficult to

analyze $(s_{\alpha,\rho})_\beta$ and $(s'_{\alpha,\rho})_\beta$ as functions in α and β. Their differentiabilities are necessarily accompanied by inhomogeneous Diophantine approximations if $\rho \notin \alpha\mathbb{Z} + \mathbb{Z}$, which are much more involved than homogeneous ones.

Let us set a family of functions $\mu_\beta : [0,1] \to \mathbb{R}$ with a parameter $\beta > 1$ by

$$\mu_\beta(x) := \Xi(x, \beta).$$

From now on, we assume that $\beta > 1$ is fixed unless stated explicitly. The next lemma is immediate to observe.

Lemma 2.1. *Let $\alpha_1, \alpha_2 \in [0,1]$. The following are equivalent.*

(a) $\alpha_1 < \alpha_2$.
(b) $s'_{\alpha_1,0} < s'_{\alpha_2,0}$ *lexicographically.*
(c) $\mu_\beta(\alpha_1) < \mu_\beta(\alpha_2)$.

Proof. The equivalence between (a) and (b) is well-known. See, e.g., [18]. If $\alpha_1 < \alpha_2$, then

$$\begin{aligned} 0 \le \sum_{n=0}^{\infty} \frac{\lceil \alpha_2 n \rceil - \lceil \alpha_1 n \rceil}{\beta^{n+1}} &= \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{\lceil \alpha_2 n \rceil - \lceil \alpha_1 n \rceil}{\beta^{n}} \\ &= \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{\lceil \alpha_2 (n+1) \rceil - \lceil \alpha_1 (n+1) \rceil}{\beta^{n+1}} \\ &\le \sum_{n=0}^{\infty} \frac{\lceil \alpha_2 (n+1) \rceil - \lceil \alpha_1 (n+1) \rceil}{\beta^{n+1}}, \end{aligned}$$

since $\beta > 1$. Therefore, $\mu_\beta(\alpha_1) \le \mu_\beta(\alpha_2)$, and here, the equality never holds by Proposition 2.2. The equivalence between (a) and (c) follows. □

We summarize, from [16], the core results on μ_β we need in this paper.

Proposition 2.2. *Let $\beta > 1$ be fixed.*

(a) *The function $\mu_\beta(x)$ is continuous at $x = \alpha_0 \in [0,1]$ if and only if α_0 is irrational. At rational α_0, $\mu_\beta(x)$ is left-continuous but not right-continuous.*
(b) *Let $\alpha_0 \in [0,1]$ be irrational. If α_0 is extremely well approximable by rationals, then $\mu_\beta(x)$ is not differentiable at $x = \alpha_0$. Otherwise, $\mu_\beta(x)$ is differentiable at $x = \alpha_0$ and $\mu'_\beta(\alpha_0) = 0$.*

A precise statement of the previous proposition requires some digression on Diophantine approximations. For further details, see [16]. The singularity of μ_β now follows from the well-known theorem on Diophantine approximations due to Khinchin. Currently, it is not clear whether μ_β is a pure jump distribution or not. We will show later that the Lebesgue-Stieltjes measure associated to μ_β has no singular continuous part. To do this, we need a closer look at mechanical words of rational and irrational slopes respectively.

Let the slope α be irrational. Then $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are called *Sturmian words*. If, in addition, $\rho = 0$, then both $s_{\alpha,0}$ and $s'_{\alpha,0}$ have an infinite suffix c_α in common:

$$s_{\alpha,0} = 0c_\alpha, \quad s'_{\alpha,0} = 1c_\alpha.$$

The word c_α is called the *characteristic word* of slope α.

If α is rational, then obviously $s_{\alpha,0}$ and $s'_{\alpha,0}$ are purely periodic. To be more precise, let $\alpha = p/q \in [0,1]$ with $\gcd(p,q) = 1$. Then the shortest period word $t_{p,q}$ (resp. $t'_{p,q}$) of $s_{\alpha,0}$ (resp. $s'_{\alpha,0}$) is said to be the *lower* (resp. *upper*) *Christoffel word*, that is to say,

$$s_{\alpha,0} = (t_{p,q})^\omega, \quad s'_{\alpha,0} = (t'_{p,q})^\omega,$$

and no shorter words have this property. Since $\gcd(p,q) = 1$, the lengths of $t_{p,q}$ and $t'_{p,q}$ are equal to q. Similarly to the case of irrational slopes, some common finite word $z_{p,q}$ appears in both $t_{p,q}$ and $t'_{p,q}$ as

$$t_{p,q} = 0z_{p,q}1, \quad t'_{p,q} = 1z_{p,q}0.$$

The word $z_{p,q}$ is called the *central word*. One notes that the central word is a palindrome — a word coinciding with the reversal of itself, e.g., 010010.

Suppose that $\alpha = p/q \in [0,1]$ is rational with $\gcd(p,q) = 1$. Since μ_β is left-continuous,

$$\mu_\beta(\alpha) = \mu_\beta(\alpha-) = (s'_{\alpha,0})_\beta = ((1z_{p,q}0)^\omega)_\beta.$$

To estimate a jump of μ_β at rational α, we should determine the value of the right limit $\mu_\beta(\alpha+)$.

Lemma 2.3. *Let $\alpha = p/q \in [0,1)$ be rational with $\gcd(p,q) = 1$. Then*

$$\mu_\beta(\alpha+) = (1(z_{p,q}10)^\omega)_\beta.$$

When $\alpha = 0$, the above equation is, by convention, understood to mean that

$$\mu_\beta(0+) = (1(z_{0,1}10)^\omega)_\beta := (1(0^\omega))_\beta = \frac{1}{\beta},$$

which is coherent to the definition of mechanical words. Let us denote by $\{t\}$ the fractional part of a real t, i.e., $t = \lfloor t \rfloor + \{t\}$. It is useful below to note that the infinite suffixes of $s_{\alpha,0}$ satisfy

$$s_{\alpha,0}(n)s_{\alpha,0}(n+1)\cdots = s_{\alpha,\{\alpha n\}}, \tag{1}$$

for any $n \in \mathbb{N}$. Likewise, a corresponding statement for upper mechanical words is true as well.

Proof of Lemma 2.3. Given $\varepsilon > 0$, choose an integer $N > q$ so that

$$\frac{1}{\beta^N(\beta-1)} < \varepsilon$$

holds. We set

$$\delta_N := \min\left\{\frac{1-\{\alpha n\}}{n} : 1 \le n \le N\right\}.$$

It is readily checked that $0 < \delta_N \le \frac{1}{N}$, and that $\delta_N \le \frac{1}{mq}$ for any $1 \le mq \le N$.

We claim that if $0 < \gamma - \alpha < \delta_N$ then $s'_{\gamma,0}$ and $1(z_{p,q}10)^\omega$ have a common prefix of length N at least. One observes that $0 < \gamma - \alpha < \delta_N$ implies, for any $1 \le n \le N$,

$$\gamma n - \alpha n + \{\alpha n\} < 1, \text{ and so, } \gamma n - \lfloor \alpha n \rfloor < 1. \tag{2}$$

In particular, since $\alpha + \frac{1}{N} < \frac{p+1}{q}$, one has $\alpha < \gamma < \alpha + \delta_N < \frac{p+1}{q}$, from which it follows that the first q letters of $s'_{\gamma,0}$ satisfy

$$s'_{\gamma,0}(0)s'_{\gamma,0}(1)\cdots s'_{\gamma,0}(q-1) = s'_{\alpha,0}(0)s'_{\alpha,0}(1)\cdots s'_{\alpha,0}(q-2)1 = 1z_{p,q}1,$$

namely, $s'_{\gamma,0}(i) = s'_{\alpha,0}(i)$ for $0 \le i \le q-2$, but $s'_{\gamma,0}(q-1) = 1$ while $s'_{\alpha,0}(q-1) = 0$. We deduce that

$$\begin{aligned}
&s'_{\gamma,0}(q)s'_{\gamma,0}(q+1)\cdots s'_{\gamma,0}(N-1)\\
&= s_{\gamma,0}(q)s_{\gamma,0}(q+1)\cdots s_{\gamma,0}(N-1) \quad (\because\ \gamma n \notin \mathbb{Z} \text{ for any } q \le n \le N)\\
&= s_{\gamma,\{\gamma q\}}(0)s_{\gamma,\{\gamma q\}}(1)\cdots s_{\gamma,\{\gamma q\}}(N-q-1) \quad (\because\ (1))\\
&= s_{\gamma,0}(0)s_{\gamma,0}(1)\cdots s_{\gamma,0}(N-q-1) \qquad (3)\\
&= s_{\alpha,0}(0)s_{\alpha,0}(1)\cdots s_{\alpha,0}(N-q-1), \qquad (4)
\end{aligned}$$

where the last two equalities rest upon the right-continuity of the floor function $\lfloor\cdot\rfloor$ and upon the definition of δ_N. In more detail,

$$\lfloor \gamma m + \{\gamma q\}\rfloor = \lfloor \gamma m\rfloor$$

holds for every $0 \le m \le N-q$. Indeed, suppose otherwise that $\lfloor \gamma m+\{\gamma q\}\rfloor > \lfloor \gamma m\rfloor$, or equivalently, $\lfloor \gamma m + \alpha q + \{\gamma q\}\rfloor > \lfloor \gamma m + \alpha q\rfloor$ ($\because \alpha q = p \in \mathbb{Z}$) for some $0 \le m \le N-q$. Since $\alpha q+\{\gamma q\} = \gamma q$, one has $\lfloor \gamma m+\gamma q\rfloor > \lfloor \gamma m+\alpha q\rfloor$. From this, we infer that

$$\lfloor \alpha(m+q)\rfloor \le \lfloor \gamma m + \alpha q\rfloor < \lfloor \gamma(m+q)\rfloor \le \gamma(m+q),$$

and thus, $\gamma(m+q) - \lfloor \alpha(m+q)\rfloor \ge 1$, which contradicts (2). The equality (3) is established. Similarly,

$$\lfloor \gamma n\rfloor = \lfloor \alpha n\rfloor$$

holds for every $0 \le n \le N$. If $\lfloor \gamma n\rfloor > \lfloor \alpha n\rfloor$ for some $0 \le n \le N$, then

$$\gamma n - \lfloor \alpha n\rfloor \ge \gamma n - \lfloor \gamma n\rfloor + 1 = 1 + \{\gamma n\},$$

which yields a contradiction establishing the equality (4). Now it follows from

$$1z_{p,q}1(0z_{p,q}1)^\omega = 1(z_{p,q}10)^\omega$$

that $s'_{\gamma,0}$ and $1(z_{p,q}10)^\omega$ have a common prefix of length N at least. Finally, one concludes that

$$\begin{aligned} 0 < \mu_\beta(\gamma) - (1(z_{p,q}10)^\omega)_\beta &= (s'_{\gamma,0})_\beta - (1(z_{p,q}10)^\omega)_\beta \\ &< \frac{1}{\beta^{N+1}} + \frac{1}{\beta^{N+2}} + \cdots = \frac{1}{\beta^N(\beta-1)} < \varepsilon. \end{aligned}$$

□

Now we are able to compute the jump $\mu_\beta(\alpha+) - \mu_\beta(\alpha)$ at each rational $\alpha = p/q$.

Theorem 2.4. *Let $\alpha = p/q \in [0,1)$ be rational with $\gcd(p,q) = 1$. Then*

$$\mu_\beta(\alpha+) - \mu_\beta(\alpha) = \frac{\beta-1}{\beta(\beta^q-1)}.$$

In particular, $\mu_\beta(0+) - \mu_\beta(0) = \frac{1}{\beta}$.

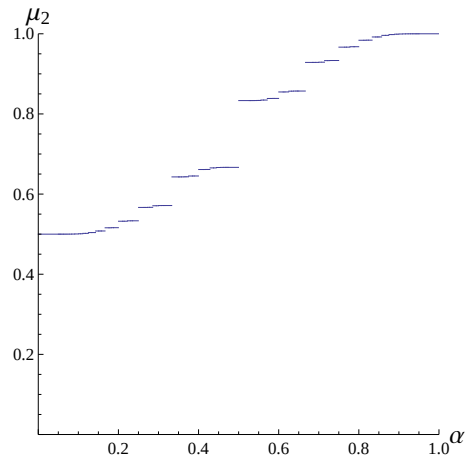

Figure 1: $\mu_2(\alpha)$

Proof. A direct computation leads us to

$$\begin{aligned}\mu_\beta(\alpha+) - \mu_\beta(\alpha) &= (1z_{p,q}1(0z_{p,q}1)^\omega)_\beta - (1z_{p,q}0(1z_{p,q}0)^\omega)_\beta \\ &= \frac{1}{\beta^q} - \frac{1}{\beta^{q+1}} + \frac{1}{\beta^{2q}} - \frac{1}{\beta^{2q+1}} + \cdots \\ &= \left(\frac{1}{\beta^q} - \frac{1}{\beta^{q+1}}\right)\left(1 + \frac{1}{\beta^q} + \frac{1}{\beta^{2q}} + \cdots\right) = \frac{\beta - 1}{\beta(\beta^q - 1)}.\end{aligned}$$

□

The graph of μ_β, when $\beta = 2$, is illustrated in Figure 1, where one can perceive a certain kind of symmetry. This is a general phenomenon as we will see below.

Let E be a morphism which rewrites each letter 0 (resp. 1) in (finite or infinite) words to 1 (resp. 0). For instance, $E(011001) = 100110$ and $E(110110\cdots) = 001001\cdots$. For brevity, we also use a notation $\overline{u} := E(u)$ for any words u. The morphism E converts mechanical words into other mechanical words.

Lemma 2.5. *Let $\alpha, \rho \in [0, 1]$. Then*

$$E(s_{\alpha,\rho}) = s'_{1-\alpha,1-\rho}, \quad E(s'_{\alpha,\rho}) = s_{1-\alpha,1-\rho}.$$

Hence, if α is irrational, then $E(c_\alpha) = c_{1-\alpha}$. As for rational $\alpha = p/q$ with $\gcd(p, q) = 1$,

$$E(z_{p,q}) = \overline{z_{p,q}} = z_{q-p,q}.$$

Proof. See [18]. □

The symmetry pattern of μ_β is formulated as follows.

Theorem 2.6. *Let* $0 \le \alpha \le 1$. *Then*

$$\mu_\beta(\alpha) + \mu_\beta(1-\alpha) = \begin{cases} \frac{2\beta-1}{\beta(\beta-1)}, & \text{if } \alpha \text{ is irrational,} \\ \frac{2\beta^q-\beta^{q-1}-\beta}{(\beta-1)(\beta^q-1)}, & \text{if } \alpha = \frac{p}{q} \text{ with } \gcd(p,q)=1. \end{cases}$$

Note that the above equation holds even when $\alpha = 0 = 0/1$ or $\alpha = 1 = 1/1$.

Proof. Suppose that α is irrational. Lemma 2.5 establishes that

$$\begin{aligned} \mu_\beta(\alpha) + \mu_\beta(1-\alpha) &= (1c_\alpha)_\beta + (1c_{1-\alpha})_\beta = \frac{2}{\beta} + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots \\ &= \frac{1}{\beta} + \frac{1}{\beta-1} = \frac{2\beta-1}{\beta(\beta-1)} \end{aligned}$$

If $\alpha = p/q$ with $\gcd(p,q) = 1$, then

$$\begin{aligned} \mu_\beta(\alpha) + \mu_\beta(1-\alpha) &= ((1z_{p,q}0)^\omega)_\beta + ((1\overline{z_{p,q}}0)^\omega)_\beta = ((\underbrace{211\cdots10}_{\text{length } q})^\omega)_\beta \\ &= \left(\frac{2}{\beta} + \frac{1}{\beta^2} + \cdots + \frac{1}{\beta^{q-1}}\right)\left(1 + \frac{1}{\beta^q} + \frac{1}{\beta^{2q}} + \cdots\right) \\ &= \frac{2\beta^q - \beta^{q-1} - \beta}{(\beta-1)(\beta^q-1)}. \end{aligned}$$

□

Since μ_β is continuous except on a countable set, we know that it is Riemann integrable. Indeed, the next integral can be evaluated thanks to the above symmetry.

Corollary 2.6.1.

$$\int_0^1 \mu_\beta(x)\ dx = \frac{2\beta-1}{2\beta(\beta-1)}.$$

Proof. Noting that $\mu_\beta(x) + \mu_\beta(1-x) = \frac{2\beta-1}{\beta(\beta-1)}$ except on a countable set, we have

$$2\int_0^1 \mu_\beta(x)\ dx = \int_0^1 \mu_\beta(x)\ dx + \int_0^1 \mu_\beta(1-x)\ dx = \frac{2\beta-1}{\beta(\beta-1)}.$$

□

3. Moments of μ_β.

With a slight abuse of notation, we also write μ_β for the Lebesgue-Stieltjes measure associated to the distribution μ_β, which hopefully causes no confusion.

The Lebesgue decomposition theorem states that any positive σ-finite measure ν is decomposed, with respect to the Lebesgue measure, in the form,

$$\nu = \nu_{ab} + \nu_{sc} + \nu_{pp},$$

where ν_{ab} is the absolutely continuous part, ν_{sc} is the singular continuous part, and ν_{pp} is the pure point part (= discrete measure).

Firstly, this section proves that μ_β (as a distribution) is a pure jump distribution, or equivalently, that μ_β (as a measure) consists only of the pure point part in its Lebesgue decomposition. We will prove this by showing that all increases in μ_β are possible by discontinuous jumps only.

Theorem 3.1. *The measure μ_β is discrete, more precisely,*

$$\mu_\beta = \sum_{\substack{0 \le p/q < 1 \\ \gcd(p,q)=1}} \frac{\beta - 1}{\beta(\beta^q - 1)} \delta_{p/q},$$

where δ_t is the Dirac measure and the summation runs over all reduced rationals in $[0, 1)$.

Proof. Since $\mu_\beta(0+) - \mu_\beta(0) = \mu_\beta(0+) = \frac{1}{\beta}$ and $\mu_\beta(1) = \frac{1}{\beta-1}$, it suffices to show the next equality by Theorem 2.4:

$$\sum_{\substack{0 < p/q < 1 \\ \gcd(p,q)=1}} \frac{\beta - 1}{\beta(\beta^q - 1)} = \frac{1}{\beta - 1} - \frac{1}{\beta} = \frac{1}{\beta(\beta - 1)}.$$

The summation uses a standard technique (cf. [6]), but we include the proof for readers' convenience. Let $\mathcal{R} := \{(p, q) : 0 < p < q \text{ and } p, q \in \mathbb{N}\}$. Then

$$\sum_{(p,q)\in\mathcal{R}} \frac{1}{\beta^q} = \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \frac{1}{\beta^j} = \sum_{k=1}^{\infty} \frac{1}{\beta^k(\beta - 1)} = \frac{1}{(\beta - 1)^2}.$$

Being computed from the other aspect, the sum is given by

$$\sum_{(p,q)\in\mathcal{R}} \frac{1}{\beta^q} = \sum_{\substack{0 < p/q < 1 \\ \gcd(p,q)=1}} \sum_{j=1}^{\infty} \frac{1}{\beta^{jq}} = \sum_{\substack{0 < p/q < 1 \\ \gcd(p,q)=1}} \frac{1}{\beta^q - 1}.$$

Gathering these, we find that

$$\sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)} = \frac{\beta-1}{\beta} \cdot \frac{1}{(\beta-1)^2} = \frac{1}{\beta(\beta-1)}.$$

□

Note that μ_β is a probability measure if and only if $\beta = 2$. Theorem 3.1 tells us that for any continuous $f : [0,1] \to \mathbb{R}$, its integration is nothing but a summation:

$$\int_0^1 f \, d\mu_\beta = \sum_{\substack{0\le p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)} f(p/q).$$

The rest of this section is devoted to the moment $M_m := \int_0^1 x^m d\mu_\beta$ of the measure μ_β. Clearly, the 0th moment is

$$M_0 = \frac{1}{\beta-1}.$$

Since $0^m = 0$, the mth moment is represented as

$$M_m = \int_0^1 x^m d\mu_\beta = \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)} \left(\frac{p}{q}\right)^m, \quad m = 1, 2, \ldots. \tag{5}$$

We will express this summation in terms of the Bernoulli numbers, which emerge from diverse contexts of mathematics.

The Bernoulli number B_n is the sequence of rational numbers defined by the power series,

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}.$$

It readily follows that $B_{2k+1} = 0$ for $k \ge 1$, and the first few nonzero terms are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \ldots.$$

The next identity is no doubt one of folklore. See, e.g., [10].

Lemma 3.2. *Let $m, n \geq 1$ be integers. Then*

$$\sum_{k=1}^{n-1} k^m = \sum_{r=0}^{m} \frac{1}{m+1-r} \binom{m}{r} n^{m+1-r} B_r.$$

Note that if $n = 1$ and $m \geq 1$, then the right-hand side vanishes, i.e., the above equality holds. On the other hand, if $m = 0$, then this equality never holds for any $n \geq 1$. We also recall that the polylogarithm $\mathrm{Li}_n(z)$ of order n is a special function defined by the power series,

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

For each integer $-1 \leq n \leq 1$, the polylogarithm can be expressed in a closed form:

$$\mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \ \mathrm{Li}_0(z) = \frac{z}{1-z}, \ \mathrm{Li}_1(z) = -\ln(1-z).$$

We are now in a position to compute the moment of μ_β. It should be noted that the calculation of (5) is a little different from that of Theorem 3.1.

Theorem 3.3. *Let M_m, $m \geq 0$ be the mth moments of μ_β. Then*

$$M_m = \frac{\beta - 1}{\beta} \sum_{r=0}^{m} \frac{1}{m+1-r} \binom{m}{r} B_r \mathrm{Li}_{r-1}(\beta^{-1}).$$

Proof. If $m = 0$, then the right-hand side is equal to

$$\frac{\beta-1}{\beta} \binom{0}{0} B_0 \mathrm{Li}_{-1}(\beta^{-1}) = \frac{1}{\beta - 1} = M_0.$$

Suppose $m \geq 1$. Let $\mathcal{R} \subset \mathbb{N}^2$ be defined as in the proof of Theorem 3.1. To calculate the moment $M_m = \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)} \left(\frac{p}{q}\right)^m$, we evaluate the sum $\sum_{(p,q)\in\mathcal{R}} \frac{(p/q)^m}{\beta^q}$ in two different ways. First, one has

$$\begin{aligned}\sum_{(p,q)\in\mathcal{R}} \frac{(\frac{p}{q})^m}{\beta^q} &= \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \sum_{j=1}^{\infty} \frac{(\frac{jp}{jq})^m}{\beta^{jq}} = \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \left(\frac{p}{q}\right)^m \sum_{j=1}^{\infty} \frac{1}{\beta^{jq}} \\ &= \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{(\frac{p}{q})^m}{\beta^q - 1}.\end{aligned}$$

On the other hand, we derive

$$\begin{aligned}\sum_{(p,q)\in\mathcal{R}} \frac{(\frac{p}{q})^m}{\beta^q} &= \sum_{j=1}^{\infty}\sum_{k=1}^{j-1}\frac{(\frac{k}{j})^m}{\beta^j} = \sum_{j=1}^{\infty}\frac{1}{j^m\beta^j}\sum_{k=1}^{j-1}k^m \\ &= \sum_{j=1}^{\infty}\frac{1}{j^m\beta^j}\sum_{r=0}^{m}\frac{\binom{m}{r}j^{m+1-r}B_r}{m+1-r} \\ &= \sum_{r=0}^{m}\frac{\binom{m}{r}B_r}{m+1-r}\sum_{j=1}^{\infty}\frac{1}{j^{r-1}\beta^j} = \sum_{r=0}^{m}\frac{\binom{m}{r}B_r}{m+1-r}\mathrm{Li}_{r-1}(\beta^{-1}).\end{aligned}$$

□

For $m \geq 1$, the formula (5) may be written as

$$M_m = \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)}\left(1-\frac{p}{q}\right)^m.$$

Consequently, one derives

$$\begin{aligned}M_m &= \sum_{\substack{0<p/q<1 \\ \gcd(p,q)=1}} \frac{\beta-1}{\beta(\beta^q-1)}\sum_{r=0}^{m}(-1)^r\binom{m}{r}\left(\frac{p}{q}\right)^r \\ &= \frac{1}{\beta(\beta-1)} + \sum_{r=1}^{m}(-1)^r\binom{m}{r}M_r.\end{aligned}$$

This enables us to express the mth moment M_m in terms of lower moments when $m \geq 1$ is odd.

Theorem 3.4. *If $m \geq 1$ is odd, then*

$$\begin{aligned}M_m &= \frac{1}{2\beta(\beta-1)} + \frac{1}{2}\sum_{r=1}^{m-1}(-1)^r\binom{m}{r}M_r \\ &= \frac{1}{(m+1)\beta(\beta-1)} + \frac{1}{m+1}\sum_{r=1}^{m-1}(-1)^r\binom{m+1}{r}M_r.\end{aligned}$$

Proof. Let $m \geq 1$ be odd. Then

$$M_m = \frac{1}{\beta(\beta-1)} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} M_r - M_m,$$

and

$$M_{m+1} = \frac{1}{\beta(\beta-1)} + \sum_{r=1}^{m-1} (-1)^r \binom{m+1}{r} M_r - \binom{m+1}{m} M_m + M_{m+1}.$$

□

4. Asymptotics of the moments of μ_β.

Since the formula given in Theorem 3.3 is an alternating sum, it gives no information on its asymptotics. To investigate the asymptotics, we first consider the exponential generating function $\mathcal{M}(x)$ of M_m:

$$\mathcal{M}(x) := \sum_{m=0}^{\infty} \frac{M_m}{m!} x^m.$$

Lemma 4.1. *The function $\mathcal{M}(x)$ is given by*

$$\mathcal{M}(x) = \frac{\beta-1}{\beta} \sum_{k=1}^{\infty} \frac{e^x - 1}{(e^{x/k} - 1)\beta^k},$$

and, as $x \to \infty$, we have

$$\mathcal{M}(x) \sim \frac{\beta-1}{\beta} (\log \beta)^{-3/4} \pi^{1/2} e^x x^{1/4} e^{-2\sqrt{x \log \beta}}.$$

Proof. By Theorem 3.3, one finds that

$$\begin{aligned}
\frac{\beta}{\beta-1}\mathcal{M}(x) &= \sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{r=0}^{m}\frac{\binom{m}{r}B_r}{m+1-r}\sum_{k=1}^{\infty}\frac{1}{k^{r-1}\beta^k} \\
&= \sum_{k=1}^{\infty}\frac{1}{\beta^k}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{x^m B_r}{k^{r-1}r!(m-r+1)!} \\
&= \sum_{k=1}^{\infty}\frac{1}{\beta^k}\sum_{r=0}^{\infty}\frac{B_r}{k^{r-1}r!}\underbrace{\sum_{m=r}^{\infty}\frac{x^m}{(m-r+1)!}}_{=\sum_{j=1}^{\infty}\frac{x^{j+r-1}}{j!}=(e^x-1)x^{r-1}} \\
&= \sum_{k=1}^{\infty}\frac{e^x-1}{\beta^k}\cdot\frac{k}{x}\underbrace{\sum_{r=0}^{\infty}\frac{B^r(\frac{x}{k})^r}{r!}}_{=\frac{x/k}{e^{x/k}-1}} = \sum_{k=1}^{\infty}\frac{e^x-1}{(e^{x/k}-1)\beta^k}.
\end{aligned}$$

Hence, as $x \to \infty$, we have

$$\frac{\beta}{\beta-1}\mathcal{M}(x) \sim e^x\sum_{k=1}^{\infty}\exp(-\frac{x}{k}-k\log\beta).$$

The Euler-Maclaurin summation formula shows that

$$\frac{\beta}{\beta-1}\mathcal{M}(x) \sim e^x\int_0^{\infty}\exp(-\frac{x}{y}-y\log\beta)\,dy = 2e^x\left(\frac{x}{\log\beta}\right)^{1/2}K_1(2\sqrt{x\log\beta}),$$

where K_1 is the modified Bessel function of the second kind, and its asymptotics is well-known (see [1, 9.7.2]). So we see that

$$\begin{aligned}
\frac{\beta}{\beta-1}\mathcal{M}(x) &\sim 2e^x\left(\frac{x}{\log\beta}\right)^{1/2}\left(\frac{\pi}{4(x\log\beta)^{1/2}}\right)^{1/2}e^{-2\sqrt{x\log\beta}} \\
&= (\log\beta)^{-3/4}\pi^{1/2}e^x x^{1/4}e^{-2\sqrt{x\log\beta}}.
\end{aligned}$$

□

Noting that M_m is a decreasing sequence, we obtain the next asymptotics.

Theorem 4.2. *The mth moment M_m satisfies*

$$M_m = O(m^{1/4}e^{-2\sqrt{m\log\beta}}) \quad \text{as } m \to \infty.$$

Proof. One derives, for every $m \geq 0$,

$$\mathcal{M}(m) = \sum_{j=0}^{\infty} \frac{M_j}{j!} m^j > \sum_{j=0}^{m} \frac{M_j}{j!} m^j > M_m \sum_{j=0}^{m} \frac{m^j}{j!} \sim M_m \frac{e^m}{2} \quad \text{as } m \to \infty.$$

As for the last asymptotics, see, e.g., [21, Problem 96]. Therefore, we have, as $m \to \infty$,

$$M_m < 2\frac{\beta - 1}{\beta} (\log \beta)^{-3/4} \pi^{1/2} m^{1/4} e^{-2\sqrt{m \log \beta}}.$$

□

Remark 4.3. Lemma 4.1 tells us that, for any $\varepsilon > 0$, there is a constant $A > 0$ such that

$$|\widetilde{\mathcal{M}}(x)| := |\mathcal{M}(x) e^{-x}| \leq A \exp((-2\sqrt{\log \beta} + \varepsilon)|x|^{1/2}).$$

Thanks to this, the 'exponential depoissonization' technique indeed works. But our case lies on the boundary of [11, Theorem 3] , i.e., the exponent of m is $1/2$. As a consequence, the error term is dominating, and so we would obtain a weaker result than Theorem 4.2.

5. Moments of μ_2.

This section demonstrates the moments of μ_β when $\beta = 2$. The mth moment for small m can be written in a rather familiar form. But the higher moments necessarily involve numerical approximation for the present. This is mainly because there are no simple formulae for higher order polylogarithms.

We recall from [17] some values of the polylogarithms $\mathrm{Li}_n(2^{-1})$.

n	$\mathrm{Li}_n(2^{-1})$
-1	2
0	1
1	$\ln 2$
2	$\frac{1}{12}\pi^2 - \frac{1}{2}(\ln 2)^2$
3	$\frac{1}{6}(\ln 2)^3 - \frac{1}{12}\pi^2 \ln 2 + \frac{7}{8}\zeta(3)$

Here, $\zeta(s)$ is the Riemann zeta function. Plugging these values into Theorem 3.3, we obtain the following table.

m	M_m	
0	1	$= 1$
1	$\frac{1}{4}$	$= 0.25$
2	$\frac{1}{12} + \frac{1}{12}\ln 2$	≈ 0.1410956
3	$\frac{1}{8}\ln 2$	≈ 0.0866434
4	$-\frac{1}{20} + \frac{1}{6}\ln 2 - \frac{1}{360}(\ln 2)^3 + \frac{1}{720}\pi^2 \ln 2 - \frac{7}{480}\zeta(3)$	≈ 0.0565710
5	$-\frac{1}{12} + \frac{5}{24}\ln 2 - \frac{1}{144}(\ln 2)^3 + \frac{1}{288}\pi^2 \ln 2 - \frac{7}{192}\zeta(3)$	≈ 0.0386885

The numerical approximations of some higher moments are

$$M_6 \approx 0.0274286,\ M_7 \approx 0.0200077,\ M_8 \approx 0.0149325,$$

$$M_9 \approx 0.0113546,\ M_{10} \approx 0.0087686,\ M_{11} \approx 0.0068603.$$

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