제 1 절 Linear Systems

1. (Manufacturing)

   한 회사는 세가지 타입의 컴퓨터를 생산하다 : Cyclone, Cyclops, Cycloid. Cyclone을 조립하기 위하여 10시간, 그 것을 테스트하기 위해 2시간 그리고 Software를 설치하기 위해 2시간이 걸리고, Cyclops는 각각 12시간, 2.5시간과 2시간이 걸린다. 또한 Cycloid는 각각 6시간, 1.5시간, 1.5시간이 걸린다.

   만약 회사가 한 달에 조립에 1,560시간, 테스트에 340시간 그리고 설치에 320시간의 노동시간을 가지고 있다면 한 달에 각 종류의 컴퓨터를 얼마나 많이 생산할 수 있는 가?

2. (Foreign Currency Exchange)

   An international business person takes fixed amounts of Japanese yen, English pounds, and German marks during each of her business trips. She trabeled three times this year. The first time she exchanged a total of \$2,550 at the following rates: the dollar was 100 yen, 0.6 pounds, and 1.6 marks. The second time she exchanged a total of \$2,840 at these rates: the dollar was 125 yen, 0.5 pounds, and 1.2 marks. The third time she exchanged a total of \$2,800 at these rates: the dollar was 100 yen, 0.6 punds, and 1.2 marks.

   How many yen, pounds, and marks did she buy each time?

3. (Inheritance) A father plans to distribute his estate, worth \$234,000, among his four daughters as follows: $\frac{2}{3}$ of the estate is to be split equally among the daughters. For the rest, each daughter is to receive \$3,000 for each year that remains until her 21st birthday.

   Given that the daughters are all 3 years apart, how much would each receive from her father's estate? How old are the daughters now?

4. (Weather) The average of the temperatures for the cities of Seoul, Gwangju and Ganglung was 88 during a given summer day. The temperature in Gwangju was 9 higher than the average of the temperature of the other two cities. The temperature in Ganglung was 9 lower than the average temperature of the other two cities.

   What are the temperature in each one of the cities?

5. (Chemical Solutions) It takes three different ingredients, A, B, and C, to produce a certain chemical substance. A, B, and C have to be dissolved in water separately before they interact to form the chemical. The solution containing A at 1.5 g per cubic centimeter (g/cm^3) combined with the solution containing B at 3.6 g/cm^3 combined with the solution containing C at 5.3 g/cm^3 makes 25.07 g of the chemical. If the proportions for A, B, C in these solutions are changed to 2.5, 4.3, and 2.4 g/cm^3, respectively (while the volumes remain the same), then 22.36 g of the chemical is produced. Finally, if the proportions are changed to 2.7, 5.5, and 3.2 g/cm^3, respectively, then 28.14 g of the chemical is produced. What are the volumes in cubic centimeters of the solutions containing A, B, and C?

6. (Balancing a Chemical Reaction)

   We need to insert integer coefficients in front of each one of the reactants so that the number of atoms of each element is the same on both sides of the equations. For example, consider the reaction of the burning of methane:

$$a\,CH_4 + b\,O_2 \quad \longrightarrow \quad c\,CO_2 + d\,H_2O$$

   Compute the coefficients a, b, c, and d that will balance the reaction.

7. (Dynamical systems : a population growth model)

Suppose we have a population of in sects divided into three age groups, A, B, C. Group A consists of insects 0-1 weeks old, group B consists of insects 1-2weeks old, and group C consists of insects 2-3 weeks old. Suppose the groups have A_k, B_k and C_k insects at the end of the kth week. We want to study how A, B, C change over time, given the following two conditions:

1. (Birth Rate) Each insect from group A has $\frac{2}{5}$ offspring, each insect from group B has 4 offspring, and each insect from group C has 5 offspring. In year $k+1$ the insects of group A are offspring of insects in year k. Hence

$$A_{k+1} = \frac{2}{5}A_k + 4B_k + 5C_k.$$

2. (Survival Rate) Only 10% of age group A survive a week.

And only 40% of age group B survive a week.

First find the dynamical system.

If the insect population starts out with 1000 from each age group, how many insects are in each group at the end of the third week?

8. (Distance from point to plane)

   A plane $\mathcal{P}$ perpendicular to a vector $\mathbf{n} = (a, b, c)$, called the normal vector of the plane, can be expressed by

$$ax + by + cz + d = 0.$$

   Find a formula for the shortest distance ℓ from the point $P(x_0, y_0, z_0)$ to the plane $\mathcal{P}$.

9. (Euclidean Geometry)

Vectors can be used to prove theorems in Euclidean geometry.

1. Prove that the line segment that bisects two sides of a triangle equals one-half of the third side.

2. Show that the diagonals of parallelogram bisect each other.

3. Show that any angle inscribed in a semicircle is a right angle.

10. (Physics and Engineering : Work done by a constant force)

If a force $\mathbf{F}$ moves an object distance $\mathbf{d}$ in the direction of $\mathbf{F}$, then the work done is

$$W = \|\mathbf{F}\|\,\|\mathbf{d}\|.$$

If $\mathbf{F}$ and $\mathbf{d}$ are at an angle θ, then W is defined as the numerical component of $\mathbf{F}$ in the direction of $\mathbf{d}$:

$$W = \|\mathbf{F}\|\,\|\mathbf{d}\|\,\cos\theta = \mathbf{F}\cdot\mathbf{d}.$$

1. Compute the work done by the constant force $\mathbf{F} = (4, -2, 1)$ if its point of application moves from $P(0, 1, -2)$ to $Q(3, 0, 1)$.

2. Compute the force $\mathbf{F}$ that has to be exerted on the rope to balance the object weighing 500kg if the inclined plane has angle of inclination $30°$. (Interpret using figure)

11. (Physics and Engineering : Moment)

The moment m of a force $\mathbf{F}$ about a point P is the product $m = \|\mathbf{F}\|\, d$, where d is the shortest distance from P to the line ℓ determined by the direction of $\mathbf{F}$.

Let Q be any point on ℓ and let $\mathbf{r} = \overrightarrow{PQ}$. Then $d = \|\mathbf{r}\|\, \sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. Therefore

$$m = \|\mathbf{r}\| \cdot \|\mathbf{F}\|\, \sin\theta = \|\mathbf{r} \times \mathbf{F}\|.$$

We define the moment vector, or torque $\mathbf{m}$, as

$$\mathbf{m} = \mathbf{r} \times \mathbf{F}.$$

A force of $3N$ at an angle of $60°$ with the positive x-axis is applied at the end of the position vector $\mathbf{r} = (\sqrt{3}, 1, 0)$.

Compute the torque of the force on $\mathbf{r}$. What is the moment about the origin?

12. (Codes)

Governments, national security agencies, companies are often interested in the transmission of coded messages that are hard to be decoded by others, if intercepted, yet easily decoded by the receiving end. There are many interesting ways of coding a messages, most of which use number theory or linear algebra. Let us discuss one that is effective, especially when a large-size invertible matrix is used.

Let M be an invertible matrix that is known only to the transmitting and receiving ends :

$$M = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}.$$

Suppose we want to code the message " **ATTACK NOW** ".

We replace each letter with the number that corresponds to the letter's position in the alphabet. We use 0 for an empty space.

$$\begin{array}{ccccccccccc} A & T & T & A & C & K & & N & O & W \\ 1 & 20 & 20 & 1 & 3 & 11 & 0 & 14 & 15 & 23 \end{array}$$

The message has now been converted into the sequence of numbers by multiplying M:

$$M \begin{bmatrix} 1 & 20 & 3 & 0 & 15 \\ 20 & 1 & 11 & 14 & 23 \end{bmatrix} = \begin{bmatrix} 77 & -56 & 35 & 56 & 47 \\ 39 & -18 & 19 & 28 & 31 \end{bmatrix}.$$

Now we have the coded message: $77, 39, -56, -18, 35, 19, 56, 28, 47, 31$.

After the receiving end receives the coded message, he needs to compute M^{-1} to decode it:

$$M^{-1} = \begin{bmatrix} -1 & 2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

Multiplying it by the coded message, he can get the original message " **ATTACK NOW** " :

$$M^{-1} \begin{bmatrix} 77 & -56 & 35 & 56 & 47 \\ 39 & -18 & 19 & 28 & 31 \end{bmatrix} = \begin{bmatrix} 1 & 20 & 3 & 0 & 15 \\ 20 & 1 & 11 & 14 & 23 \end{bmatrix}.$$

13. (Coding Problem)

Problem A. (Decoding Message)

Decode the message given by the numbers 17, 15,29, 15, 17, 29, 16, 31, 47, 6, 19, 20, 35, 24, 39, 14, 19, 19 if the coding matrix is given by

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Problem B. (Code Breaking)

Suppose that you intercepted the following coded stock market message:

$$1156, -203, 624, -84, -228, 95, 1100, -165, 60, 19.$$

Your sources inform you that the message was coded by using a 2×2 symmetric matrix. Your intuition tells you that the first word of the message is very likely to be either sell or buy.
Can you break the code?

14. (Initial Value Problem)

격리된 인구집단 $p(t)$에서의 인구증가율이 $f(p)$라 하자.

A. $f(p) = a\,p$ $(a > 0)$ 이고, $p(0) = p_0$ 일때 $p(t)$를 구하여라.

B. 인구 $p(t)$의 초만원 상태가 사망률의 증가와 출생률의 감소를 초래할 경우에는 $f(p) = ap - bp^2$ $(a, b > 0)$로 주어질 수 있다. $p(0) = p_0$ 일때, $p(3)$를 구하여라. 단, $\Delta t = 1$로 하고 Euler 법을 사용하라.