Vector Calculus

Leture Note

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Contents

1 Differentiation and Line Integrals

Definition (Derivative). Let U be an open set in $\mathbb{R}^n$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is **differentiable** at $\mathbf{x_0} \in U$ if the partial derivatives of f exist at $\mathbf{x_0}$ and if

$$\lim_{\mathbf{x} \to \mathbf{x_0}} \frac{||f(\mathbf{x}) - f(\mathbf{x_0}) - T(\mathbf{x} - \mathbf{x_0})||}{||\mathbf{x} - \mathbf{x_0}||} = 0$$

whrer T is the matrix with matrix elements $(\frac{\partial f_i}{\partial x_j})$ evaluated at $\mathbf{x_0}$. We call T the **derivative** of f at $\mathbf{x_0}$ and denote by $Df(\mathbf{x_0})$.

$$Df(\mathbf{x_0}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_1} \end{bmatrix}.$$

Definition (Gradient). Consider the special case $f : U \subset \mathbb{R}^n \to \mathbb{R}$.

$$Df(\mathbf{x}) = \nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

called the **gradient** of f on $\mathbf{x}$.

Definition (Directional derivative). Consider the special case $f : U \subset \mathbb{R}^n \to \mathbb{R}$.

$$\frac{d}{dt} f(\mathbf{x} + t\mathbf{v}) \mid_{t=0} = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

called the **directional derivative** of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{v}$ if it exists.

Theorem 1.1 (Directional derivative and Gradient).

$$\begin{aligned} \frac{d}{dt} f(\mathbf{x} + t\mathbf{v}) \mid_{t=0} &= Df(\mathbf{x})\,\mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v} \\ &= \frac{\partial f(\mathbf{x})}{\partial x_1} v_1 + \frac{\partial f(\mathbf{x})}{\partial x_2} v_2 + \cdots + \frac{\partial f(\mathbf{x})}{\partial x_n} v_n \end{aligned}$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ *and* $||\mathbf{v}|| = 1$.

1. Assume $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then $\nabla f(\mathbf{x})$ points in the direction along which f is increasing the fastest.

2. $\nabla f(\mathbf{x_0})$ is normal to the level surface:

Let S be the surface consisting of those (x, y, z) such that $f(x, y, z) = k$. The **tangent plane** of S at a point (x_0, y_0, z_0) of S is defined by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$.

Definition (Path Integrals). The **path integral** or the **integral of** $f(x, y, z)$ **along the path** σ is defined when $\sigma : I = [a, b] \to \mathbb{R}^3$ is C^1 and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on I. We define this integral by the equation

$$\begin{aligned}\int_\sigma f\,ds &= \int_a^b f(x(t), y(t), z(t))\,||\sigma'(t)||\,dt \\ &= \int_a^b f(\sigma(t))\,||\sigma'(t)||\,dt.\end{aligned}$$

Definition (Work done by F). Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$, continuous on the C^1 path $\sigma : [a, b] \to \mathbb{R}^3$. We define the **line integral** of $\mathbf{F}$ along σ, by the formula

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\sigma(t)) \cdot \sigma'(t) dt$$

that is, we integrate the dot product of $\mathbf{F}$ with σ' over the interval $[a, b]$.

Another case : Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ and let C be a smooth curve with position vector $\mathbf{s}$ at P. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (\mathbf{F} \cdot \mathbf{u}) ds$$

where $\mathbf{u}$ is a unit tangent vector to C at P, i.e.,

$$d\mathbf{s} = \frac{d\mathbf{s}}{ds} ds = \mathbf{u} ds.$$

Theorem 1.2. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is C^1 and that $\sigma : [a, b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then*

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a)).$$

Another case : *For any curve C joining the points P_0, P_1,*

$$\int_C \nabla f \cdot d\mathbf{s} = \int_{P_0}^{P_1} \nabla f \cdot d\mathbf{s} = f(P_1) - f(P_0).$$

Example. Let $\sigma(t) = (t^4/4, \sin^3(t\pi/2), 0)$, $t \in [0,1]$. Evaluate

$$\int_\sigma y\,dx + x\,dy.$$

Sol. $f(x,y,z) = xy$ implies $\nabla f = (y, x, 0)$.

$$\int_\sigma y\,dx + x\,dy = \int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(1)) - f(\sigma(0)) = \frac{1}{4}.$$

□

Example. Let $\mathbf{F} = (yx, xy, xz)$ and let C be a curve consisting of the curve $x = y^2$, $z = 0$ in the xy plane from $(1,1,0)$ to $(1,1,0)$.

Evaluate the work done by the force in moving the particle along C.

Sol.

$$x = y^2, \quad z = 0 \quad \Longrightarrow dx = 2y\,dy, \quad dz = 0,$$

$$\mathbf{F} \cdot d\mathbf{s} = yzdx + xydy + xzdz = y^3 dy.$$

$$\int_C \mathbf{F} \cdot \, d\mathbf{s} = \int_0^1 y^3\,dy = \frac{1}{4}.$$

□

2 Vector Fields

Definition (Vector Field). A **vector field** on $\mathbb{R}^n$ is a map $\mathbf{F} : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\mathbf{x}$ in its domain A a vector $\mathbf{F}(\mathbf{x})$.

Definition (Flow line). If $\mathbf{F}$ is a vector field, a **flow line** for $\mathbf{F}$ is a path $\sigma(t)$ such that

$$\sigma'(t) = \mathbf{F}(\sigma(t)).$$

That is, $\mathbf{F}$ yields the velocity field of the path $\sigma(t)$.

Remark. Analytically, the problem of finding a flow line that passes through $\mathbf{x_0}$ at time $t = 0$ involves solving the differential equation with initial condition:

$$\sigma'(t) = \mathbf{F}(\sigma(t)); \quad \sigma(0) = \mathbf{x_0}.$$

Definition (Flows of Vector Fields). We call the mapping ϕ the **flow** of $\mathbf{F}$ when $\phi(\mathbf{x}, t)$ is defined by

$$\phi(\mathbf{x}, t) = \left\{ \begin{array}{l} \text{the position of the point on the flow line} \\ \text{through } \mathbf{x} \text{ after time } t \text{ has elapsed} \end{array} \right\}$$

which satisfies

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t))$$
$$\phi(\mathbf{x}, 0) = \mathbf{x}.$$

Remark.

$$\frac{\partial}{\partial t} D_{\mathbf{x}}\phi(\mathbf{x}, t) = D_{\mathbf{x}}\mathbf{F}(\phi(\mathbf{x}, t)) \cdot D_{\mathbf{x}}\phi(\mathbf{x}, t)$$

is called the **equation of first variation**.

Theorem 2.1 (Curl of gradient and Divergence of curl).

For any C^2 function f we have

$$\nabla \times (\nabla f) = \mathbf{0}.$$

For any C^2 vector field $\mathbf{F}$ we have

$$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

$\nabla \times \mathbf{F}$ is related to rotations

$\nabla \times \mathbf{F} = \mathbf{0} \Longrightarrow \mathbf{F}$ is irrotational

$\nabla \cdot \mathbf{F}$ is related to compressions and expansions

$\nabla \cdot \mathbf{F} = 0 \Longrightarrow \mathbf{F}$ is incompressible

3 Green's Theorem [Divergence Theorem]

Theorem 3.1 (Green's Theorem). *Let D be a region on $\mathbb{R}^2$ and let C be its boundary. Suppose $P :\longrightarrow \mathbb{R}$ and $Q :\longrightarrow \mathbb{R}$ are C^1. Then*

$$\int_{C^+} P\,dx + Q\,dy \;=\; \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Proof.

$$\frac{\partial Q}{\partial x} dxdy \quad \text{on} \quad D \;=\; Q\,dy \quad \text{on} \quad C^+, \qquad \frac{\partial P}{\partial y} dxdy \quad \text{on} \quad D \;=\; -\,P\,dx \quad \text{on} \quad C^+.$$

□

Example. For $P(x, y) = x$ and $Q(x, y) = xy$ where D is the unit disc $x^2 + y^2 \leq 1$.

$$\int_{\partial D} Pdx + Qdy = \int_0^{2\pi} [(\cos t)(-\sin t) + \cos t \sin t \cos t]dt = 0.$$

So

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_D ydxdy = 0.$$

Corollary 3.2. *If C is a simple closed curve that bounds a regin to which Green's Theorem applies, then the area of the region D bounded by C is*

$$A = \frac{1}{2} \int_{\partial D} xdy - ydx.$$

Since

$$\frac{1}{2} \int_{\partial D} xdy - ydx = \frac{1}{2} \int_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy = \int_D dxdy = A.$$

Theorem 3.3 (Vector Form of Green's Theorem). *Let $D \subset \mathbb{R}^2$ be a region and let ∂D be its boundary. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a C^1 vector field on D. Then*

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_D (\mathrm{curl}\mathbf{F}) \cdot \mathbf{k}\, dA = \int_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

where

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}.$$

Example. Let $\mathbf{F} = (xy^2, y + x)$. Integrate $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ over the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$.

Sol. The First Case :

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = (1 - 2xy)\mathbf{k} \quad \Longrightarrow \quad (\nabla \times \mathbf{F}) \cdot \mathbf{k} = 1 - 2xy.$$

Hence

$$\begin{aligned}\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy &= \int_0^1 \int_{x^2}^{x} (1 - 2xy)\, dydy \\ &= \int_0^1 [y - xy^2]_{x^2}^{x}\, dx \\ &= \int_0^1 [x - x^3 - x^2 + x^5]\, dx = \frac{1}{12}.\end{aligned}$$

The Second Case :

With $C_1 \ : \ y = x$ and $C_2 \ : \ y = x^2$,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1 \cup C_2} F_1\, dx + F_2\, dy.$$

$$\begin{aligned}\int_{C_1} F_1 dx + F_2 dy &= \int_0^1 (xy^2)dx + (y + x)dy = \int_0^1 (xx^2)dx + (x + x)d(x) \\ &= \int_0^1 (x^3 + 2x)dx = \frac{5}{4} \\ \int_{C_2} F_1 dx + F_2 dy &= \int_0^1 (xy^2)dx + (y + x)dy = \int_0^1 (x(x^2)^2)dx + (x + x^2)d(x^2) \\ &= \int_0^1 (x^5 + 2x^2 + 2x^3)dx = \frac{4}{3}.\end{aligned}$$

□

Theorem 3.4 (Divergence Theorem in the Plane). *Let $D \subset \mathbb{R}^2$ be a region and let ∂D be its boundary. Let $\mathbf{n}$ denote the outward unit normal to ∂D, which is given by*

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \left(\frac{dy}{d\sigma}, -\frac{dx}{d\sigma}\right).$$

if $\sigma : [a, b] \longrightarrow \mathbb{R}^2, \quad t \mapsto \sigma(t) = (x(t), y(t))$ is positively oriented parametrization of ∂D.

Let $\mathbf{F} = (P, Q)$ *be a* C^1 *vector field on* D*. Then*

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \int_D \nabla \cdot \mathbf{F} \, dA.$$

since $$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{\partial D} P \, dy - Q \, dx = \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy = \int_D \operatorname{div} \mathbf{F} \, dA.$$

For the 3D-case :

$$\int_{\mathcal{R}} \Delta\varphi \, dV = \int_{\mathcal{R}} \nabla \cdot \nabla\varphi \, dV = \int_S \nabla\varphi \cdot d\mathbf{S} = \int_S \nabla\varphi \cdot \mathbf{n} \, dS = \int_S \frac{\partial\varphi}{\partial \mathbf{n}} \, dS.$$

4 Stokes' Theorem

Let S be a surface given by $z = \phi(x, y)$ on D.

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,z)\,\sqrt{1+\phi_x^2+\phi_y^2}\, dxdy$$

since

$$dS = \left|\frac{1}{\cos\gamma}\right| dxdy = \sqrt{1+\phi_x^2+\phi_y^2}\, dxdy \quad \text{with} \quad \cos\gamma = \frac{-1}{\sqrt{\phi_x^2+\phi_y^2+1^2}}.$$

where $\cos\gamma$ is the $z-$component of unit normal of $S : \Phi(x, y, z) = z - \phi(x, y)$.

[Change of variables]

$$x = f(u, v),\ y = g(u, v) \ : \ \mathcal{R} \mapsto \mathcal{R}'$$

$$\iint_{\mathcal{R}} F(x,y)\, dxdy \ = \ \iint_{\mathcal{R}'} F(f(u,v), g(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dxdy.$$

Theorem 4.1 (Surface integrals of vector field). *Let $\mathbf{F}$ be a vector field on a surface S : $z = \phi(x, y)$ on D. Then*

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{S} &= \int_S \mathbf{F}\cdot\mathbf{n}\, dS \\ &= \int_D [F_1(-z_x) + F_2(-z_y) + F_3]\,\sqrt{1+z_x^2+z_y^2}\, dxdy.\end{aligned}$$

Theorem 4.2 (Stokes' Theorem for Graphs). *Let S be the oriented surface defined by a C^2 function $z = f(x, y)$, $(x, y) \in D$, and let $\mathbf{F}$ be a C^1 vector field on S. Then if ∂S denotes the oriented boundary curve of S as defined above we have*

$$\int_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_{\partial S} \mathbf{F}\cdot d\mathbf{s}.$$

Thus Stokes' Theorem says that the integral of the normal component of the curl of a vector field $\mathbf{F}$ over a surface S is equal to the integral of the tangential component of $\mathbf{F}$ around the boundary ∂S.

1. In general $\int_C \mathbf{F}\cdot d\mathbf{s}$, being the integral of the tangential component of $\mathbf{F}$, represents the net amount of turning of the fluid in a counterclockwise direction around C.

2. One therefore refers to $\int_C \mathbf{F} \cdot d\mathbf{s}$ as the circulation of $\mathbf{F}$ around C.

3. curl $\mathbf{F}(P) \cdot \mathbf{n}$ is the circulation of $\mathbf{F}$ per unit area on a surface perpendicular to $\mathbf{n}$

4. Ovserve that the magnitude of curl $\mathbf{F} \cdot \mathbf{n}$ is maximized when $\mathbf{n} = \text{curl } \mathbf{F}/||\text{curl } \mathbf{F}||$.

5. Therefore the rotating effect at P is greatest about the axis parallel to curl $\mathbf{F}/||\text{curl } \mathbf{F}||$.

6. Thus curl $\mathbf{F}$ is aptly called the *volticity vector*.

5 Conservative Fields

Theorem 5.1 (Conservative vector Fields). *Let* $\mathbf{F}$ *be a* C^1 *vector field defined on* $\mathbb{R}^3$ *except possibly for a finite number of points. The following conditions on* $\mathbf{F}$ *are all equivalent:*

(*i*) *For any oriented simple closed curve* C, $\int_C \mathbf{F} \cdot d\mathbf{s} =$. *So* $\int_D \nabla \cdot \mathbf{F} \, dA = 0$.

(*ii*) *For any two oriented simple curves* C_1, C_2 *with the same endpoints,* $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.

(*iii*) $\mathbf{F}$ *is the gradient of some function ;* ($\mathbf{F} = \nabla f$)

(*iv*) $\nabla \times \mathbf{F} = \mathbf{0}$. ($\nabla \cdot F = \mathbf{0} \longrightarrow \nabla \times F = \mathbf{0}$)

Example. Consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) = (y,\ x\cos yz + x,\ y\cos yz).$$

Show that $\mathbf{F}$ is irrotational and find a scalar potential for $\mathbf{F}$.

Sol. **1.** Clearly, $\nabla \times \mathbf{F} = \mathbf{0}$.

2. By setting

$$f(x, y, z) = \int_0^x F_1(t, 0, 0)dt + \int_0^y F_2(x, t, 0)dt + \int_0^z F_3(x, y, t)dt,$$

one can show that $\mathbf{F} = \nabla f$. □

Theorem 5.2. *If* $\mathbf{F}$ *is a* C^1 *vector field on* $\mathbb{R}^3$ *with* div $\mathbf{F} = 0$, *then there exists a* C^1 *vector field* $\mathbf{G}$ *with* $\mathbf{F} = \text{curl } \mathbf{G}$.

6 Gauss' Theorem

Theorem 6.1 (Gauss' Divergence Theorem in 3D). *Let Ω be a region in $\mathbb{R}^3$. Denote by $\partial\Omega$ the oriented closed surface that bounds Ω. Let $\mathbf{F}$ be a smooth vector field defined on Ω. Then*

$$\int_\Omega (\nabla \cdot \mathbf{F})\, dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

or alternatively

$$\int_\Omega (\text{div } \mathbf{F})\, dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n})\, dS.$$

Example. Use the Divergence Theorem to evaluate

$$\int_{\partial W} (x^2 + y + z)\, dS$$

where W is the solid ball $x^2 + y^2 + z^2 \leq 1$.

Sol. We must find some vector field $\mathbf{F} = (F_1, F_2, F_3)$ on W with

$$\mathbf{F} \cdot \mathbf{n} = x^2 + y + z.$$

At any point $(x, t, z) \in \partial W$ the outward unit normal $\mathbf{n}$ to ∂W is $\mathbf{n} = (x, y, z)$. Therefore, from an equation

$$\mathbf{F} \cdot \mathbf{n} = F_1 x + F_2 y + F_3 z = x^2 + y + z$$

we set and solve for F_1 , F_2, F_3 to find that $\mathbf{F} = (x, 1, 1)$ and div $\mathbf{F} = 1 + 0 + 0 = 1$.

Thus by Gauss' Divergence Theorem

$$\int_{\partial W} (x^2 + y + z)\, dS = \int_W dV = \text{volume}(W) = \frac{4}{3}\pi.$$

$\square$

7 Application to 2D Incompressible Fluid Flow

If the flow is divergenceless and irrotational, (that is, if there are no distributions of sources or sinks or of vortices, about which the fluid tends to rotate, and if also the fluid is assumed to be incompressible) we have seen that the velocity vector $\mathbf{V}$ is the gradient of a function φ, called the **velocity potential**, and that φ satisfies Laplace's equation.

That is, for such a $\mathbf{V}$, $\exists\ \varphi$ s.t. $\mathbf{V} = \nabla\varphi$ and $\Delta\varphi = 0$.

- The **equipotential lines** : the level curve $\varphi(x, y) = c_1$.
- The **streme lines** : the velocity vectors of the streamelines $\psi(x, y) = c_2$ are normal to equipotential lines $\varphi(x, y) = c_1$, that is,

$$\nabla\varphi \cdot \nabla\psi = 0$$

Example. For the velocity vector $\mathbf{V} = (2x, -2y)$,

- there exists a velocity potential $\varphi = x^2 - y^2$ by using integration.
- the equipotential curves in the xy plane are the hyperbolas $x^2 - y^2 = c_1$.
- from the equation
$$\nabla\psi = \left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}\right) = (2y, 2x),$$
we can determine the stremelines which is the hyperbolas $\psi = 2xy = c_2$.
- considering $\psi = 2xy$ as the velocity potential in a conjugate flow yields that $\varphi = x^2 - y^2$ can be considered the stream function.