미분방정식 및 실습 : Final. Exam. 2006. 12.

1. a) Find the general solution of the following underdamping($c^2 < 4mk$) free vibration system

$$my'' + cy' + ky = 0.$$

(Ans) $m\lambda^2 + c\lambda + k = 0 \to \lambda = \alpha \pm i\beta$ with $\alpha = -c/2m$, $\beta = \sqrt{-(c^2 - 4mk)}/2m$
$\to \quad y = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$

b) With the coefficients $m = 1$, $k = 2$ and $c = 2$, find the general solution for the following forced vibration system

$$my'' + cy' + ky = \mathrm{e}^{-t}\cos t.$$

(Ans) $y_h = e^{-t}(C_1 \cos t + C_2 \sin t)$, $\quad y_p = (1/2)\, t\, e^{-t} \sin t \to y = y_h + y_p$

2. Find the fundamental solution $y_1(x)$ and $y_2(x)$ using the power series solution method at $x = 0$ for the following problem:

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

(Ans) $y_1 = \sum_{n=0}^{\infty} \frac{1}{2n-1} x^{2n} = -1 + x^2 + \frac{1}{3}x^4 + \cdots$, $\quad y_2 = x$

3. For the following Euler's equation with a singularity at $x = 0$

$$x^2y'' + \alpha xy' + \beta y = 0, \quad (x > 0),$$

find the general solution when $(\alpha - 1)^2 - 4\beta < 0$.

(Ans) $y = x^r$, $r = \mu \pm i\nu$ with $\mu = -\frac{\alpha-1}{2}$, $\nu = \frac{\sqrt{-(\alpha-1)^2+4\beta}}{2}$
$\to y = x^\mu \left(C_1 \cos(\nu \log x) + C_2 \sin(\nu \log x)\right)$

4. Find the Laplace transforms or inverse Laplace transform:

$$a) \quad \mathcal{L}\{e^{-t}\sin\omega t\} \qquad b) \quad \mathcal{L}^{-1}\left\{\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}\right\}$$

(Ans) a) $\mathcal{L}\{e^{-t}\sin\omega t\} = \frac{\omega}{(s+1)^2+\omega^2}$
b) $\mathcal{L}^{-1}\{F'(s)\} = -tf(t) \to \quad \mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\} = \mathcal{L}^{-1}\left\{-\left(\frac{s}{s^2+\omega^2}\right)'\right\} = t\cos\omega t$

5. Using the Laplace transform, solve

$$y'' - y' - 2y = f(t), \quad y(0) = 5, \quad y'(0) = -1,$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2, \\ 0 & \text{if } t \ge 2. \end{cases}$$

(Ans) $f(t) = 1 - u_2(t)$, $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$

$\rightarrow \quad \frac{1}{(s+1)(s-2)}Y(s) = (5s-6) + \frac{1}{s}\left(1 - e^{-2s}\right)$

$\rightarrow \quad Y(s) = \frac{11}{3(s+1)} + \frac{4}{3(s-2)} + \left(-\frac{1}{2s} + \frac{1}{3(s+1)} + \frac{1}{6(s-2)}\right)\left(1 - e^{-2s}\right)$

$\rightarrow \quad y(t) = -\frac{1}{2} + 4e^{-t} + \frac{3}{2}e^{2t} + u_2(t)\left(\frac{1}{2} - \frac{1}{3}e^{-(t-2)} - \frac{1}{6}e^{2(t-2)}\right)$

6. Using Laplace transform of convolution, solve the following integro differential equation

$$y(t) = \sin 2t + \int_0^t y(u)\,\sin 2(t-u)\,du.$$

(Ans) $Y(s) = \frac{2}{s^2+4} + Y(s)\frac{2}{s^2+4} \rightarrow y(t) = \sqrt{2}\,\sin\sqrt{2}t$