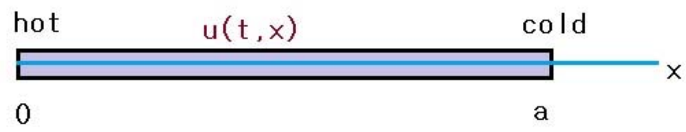
< Heat equation >

열흐름(flow of heat)에 대한 초기치 경계문제

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

$$u(0,x) = f(x)$$

$$u(t,0) = b_1, \quad u(t,a) = b_2$$



$u(t,x)$ 는 시간 t에서 위치 x 에서의 온도

β는 균질 막대의 열확산 계수

일시적으로 주어진 과도현상이 사라진 후 온도의 변화

$$u(t,x) \to b_1 + \left(\frac{x}{a}\right)(b_2 - b_1) \quad as \quad t \to \infty$$

- 시간에 대한 미분 (Euler Method)

$$0 = t_0 < t_1 < t_2 < \quad \cdots \quad < t_N = T,\ t_i = i\,h,\ h = \Delta t = \frac{T}{N}$$

$$\frac{\partial u(t_i, x)}{\partial t} \approx \frac{u(t_i, x) - u(t_{i-1}, x)}{h}$$: 후방차분법(backward difference)

For $i = 1, 2, \cdots, N$

$$\frac{u(t_i, x) - u(t_{i-1}, x)}{h} = \beta \frac{\partial^2 u(t_i, x)}{\partial x^2}$$: $t = t_i$에서 근사방정식

- 위치 x에 대한 미분(유한차분법 : Finite difference method)

$$0 = x_0 < x_1 < x_2 < \cdots < x_M = T, \; x_j = jk, \; k = \Delta x = \frac{a}{M}$$

$$\frac{\partial u(t_i, x_{j+1})}{\partial x} \approx \frac{u(t_i, x_{j+1}) - u(t_i, x_j)}{k}$$

$$\frac{\partial u(t_i, x_j)}{\partial x} \approx \frac{u(t_i, x_j) - u(t_i, x_{j-1})}{k}$$

=> $$\frac{\partial^2 u(t_i, x_j)}{\partial x^2} \approx \frac{\frac{\partial u(t_i, x_{j+1})}{\partial x} - \frac{\partial u(t_i, x_j)}{\partial x}}{k}$$

$$\approx \frac{u(t_i, x_{j+1}) - 2u(t_i, x_j) + u(t_i, x_{j-1})}{k^2}$$

$(t,x)=(t_i, x_j)$에서 근사방정식

For $i=1,2,\cdots,N,\quad j=1,2,\cdots,M-1$

$$\frac{u(t_i,x_j)-u(t_{i-1},x_j)}{h}=\beta\frac{u(t_i,x_{j+1})-2u(t_i,x_j)+u(t_i,x_{j-1})}{k^2}$$

Set $\delta=\dfrac{h}{k^2}$ and denote $u_{i,j}=u(t_i,x_j)$, $f_j=f(x_j)$

$$-\beta\delta u_{i,j-1}+(1+2\beta\delta)u_{i,j}-\beta\delta u_{i,j+1}=u_{i-1,j}\quad,\ j=2,3,\cdots,M-2$$

Case of j=1 : 왼쪽경계조건 $u_{i,0}=u(t_i,0)=b_1$

$$(1+2\beta\delta)u_{i,1}-\beta\delta u_{i,2}=u_{i-1,1}+\beta\delta b_1$$

Case of j=M-1 : 오른쪽경계조건 $u_{i,M}=u(t_i,a)=b_2$

$$-\beta\delta u_{i,M-2}+(1+2\beta\delta)u_{i,M-1}=u_{i-1,M-1}+\beta\delta b_2$$

Set

$A = \mathrm{diag}(\ -\beta\,\delta,\ \ 1+2\,\beta\,\delta,\ \ -\beta\,\delta\ )$: 삼중대각 계수행렬

$\boldsymbol{u}_i = (u_{i,1}, u_{i,2}, \cdots, u_{i,M-1}\ )^t$: 시간 t_i에서 온도 벡터

$\boldsymbol{u}_0 = (f_1, f_2, \cdots, f_{M-1}\ )^t$: 초기 시간에서의 온도 분포

$\boldsymbol{b} = (\beta\,\delta\,b_1, 0, \cdots, 0, \beta\,\delta\,b_2\ )^t$: 초기 시간에서의 온도 분포

행렬방정식

$$A\,\boldsymbol{u}_i = \boldsymbol{u}_{i-1} + \boldsymbol{b}\,,\quad i = 1, 2, \cdots, N.$$

풀이법 : 직접법(Gauss-소거법), 반복법

Movie작성 : for 문에서 M(j) = getframe; end; movie(M)

movie2avi(M,'name.avi')